Problem 1: Basic thermodynamics of an ideal gas (21 pts)

Consider the process shown in the $P - V$ diagram below for a monatomic ideal gas that undergoes an expansion from point $A$ to $B$ along a straight line path, followed by a constant pressure expansion from point $B$ to $C$, followed by a rapid expansion from point $C$ to $D$, followed by a constant temperature compression from point $D$ to point $E$, followed by a constant volume path from point $E$ back to point $A$ again. The gas consists of $N = 6 \times 10^{24}$ molecules, $P_A = 3 \times 10^4$ Pa, $V_A = 1 \text{ m}^3$, and so on as shown in the diagram. As always, please show your work and/or explain your reasoning for every part of the problem to get full credit.

(a) (3 pts) What is the temperature of the gas at point $A$?

(b) (3 pts) How much work is done by the gas along the initial expansion from point $A$ to point $B$?

(c) (3 pts) How much heat is exchanged with the gas along the path from point $B$ to $C$? State whether heat is added to or removed from the gas.

(d) (3 pts) What is the change in the internal energy of the gas as it follows the path $C \rightarrow D \rightarrow E$?

(e) (3 pts) What is the work done by the gas as it goes from point $E$ to point $A$?

(f) (3 pts) What is the change in the internal energy of the gas as it follows the path $E \rightarrow A \rightarrow B \rightarrow C \rightarrow D$?

(g) (3 pts) What is the change in the entropy of the gas resulting from the path $D \rightarrow E \rightarrow A \rightarrow B \rightarrow C$?
Problem 2: Elasticity of a polymer — entropic forces (20 pts)
(Adapted from Kittel & Kroemer Problem 3.10)

The first law of thermodynamics for a one-dimensional system can be written as

$$dU = TdS + FdL,$$

where $F$ is the external force acting on the linear system and $L$ is the length of the system. Note that the force is defined here so that extending the system increases the work done on the system, which is appropriate for the model considered below.

A simple model for a polymer consists of $N$ links, each of length $\rho$, immersed in a heat bath at temperature $T$. The polymer is freely jointed, so that each link is free to point to the left or right along the $x$-axis with no difference in the internal energy of the system. One end of the polymer is held in place at position $x_o = 0$. You may assume $N$ is even for the entire problem.

(a) (5 pts) If no external force is applied, what is the mean position $x$ of the free end of the polymer? (Note that this is not necessarily the same as the mean length of the polymer.) Defend your answer with a calculation or physical argument.

(b) (5 pts) Show that the number of arrangements that result in the position of the free end of the polymer to be $x = 2s\rho$ is

$$\Omega(N, s) = \frac{N!}{(\frac{1}{2}N + s)! (\frac{1}{2}N - s)!}.$$

(c) (5 pts) For $|s| \ll N$, show that the entropy as a function of $x$ is

$$S(x) \approx k_B N \ln 2 - \frac{k_B x^2}{N \rho^2}.$$

(d) (5 pts) What external force is required to hold the free end of the polymer at position $x$? Express your answer as a function of $x$, $T$, $\rho$, $N$, and $k_B$. 

Problem 3: Statistics of effused particles (24 pts)
(Adapted from Pathria Section 6.4)

Consider a large box containing an ideal gas of $N$ monatomic particles in thermal equilibrium at temperature $T$ and pressure $P$. Use the axis directions indicated in the diagram.

(a) (4 pts) What is the mean kinetic energy per molecule of the particles in the box?

(b) (5 pts) What is the distribution of velocities of the gas particles inside the box? Express your answer as a function of the three components of the velocity, $P(v_x, v_y, v_z)$.

(c) (5 pts) There is a very small hole of area $A$ in the left wall of the box through which some particles escape, or effuse. Calculate the distribution of velocities of the particles that are effusing from the box during a short time $t$. A good qualitative description of how and why the effusing particles’ velocity distribution differs from that of the particles in the box is worth half the credit of the full-blown calculation.

(d) (5 pts) What is the mean kinetic energy per molecule of the effusing particles?

(e) (5 pts) After particles have been escaping for a short time $t$, what is the total momentum carried by the effused particles?
Problem 4: Fermi gas (15 pts)
(Adapted from Kittel & Kroemer example starting on p. 183)

Consider a gas of \( N \) noninteracting electrons confined to a cube-shaped box of volume \( V = L^3 \), where \( L \) is the length of any side.

(a) (5 pts) What is the Fermi energy \( E_F \) of the system? Express your answer in terms of \( N, V, \) the electron mass \( m, \) and fundamental constants.

(b) (5 pts) Show that the density of states \( D(E) \) is

\[
D(E) = \frac{V}{2\pi^2} \left( \frac{2m}{\hbar^2} \right)^{3/2} E^{1/2},
\]

where \( D(E)dE \) is the number of orbitals with kinetic energy between \( E \) and \( E + dE \).

(c) (5 pts) What is the kinetic energy of the ground state of the system? Express your answer in terms of \( N, V, m \) and fundamental constants.
**Problem 5: Isentropic expansion of a photon gas (25 pts)**

(Adapted from Kittel & Kroemer Problems 4.17 & 4.18)

We can model the cosmic black-body radiation in our universe as a gas of photons of thermal equilibrium radiation in a cube of volume $V$ at temperature $T$. As the cube expands, the radiation pressure performs positive work and the temperature drops.

(a) (5 pts) Remembering that each mode of the photon gas behaves as a harmonic oscillator, derive a closed form expression for the partition function $Z$ for one mode of the electromagnetic radiation at frequency $\omega$. Express your answer as a function of $\omega$, $\hbar$, $T$, and Boltzmann’s constant $k$.

(b) (5 pts) Show that the occupancy, or expected number of photons, of any one mode is given by

$$<s> = \frac{1}{\exp(\hbar \omega / kT) - 1}.$$

(c) (5 pts) Show that the entropy per mode in the gas can be written as

$$S = k <s + 1> \ln <s + 1> - k <s> \ln <s>.$$

(d) (5 pts) From part c, we deduce that the entropy will not change during the expansion provided the occupancy of each mode does not change, which we will assume for the rest of this problem. Given that the entropy $S$ and energy $U$ of a photon gas is given by

$$S = \frac{4\pi^2 kV}{45} \left(\frac{kT}{\hbar c}\right)^3,$$

$$\frac{U}{V} = \frac{\pi^2}{15\hbar^3c^3}(kT)^4,$$

what was the radius of our model universe when the radiation was at 3000K, compared to the current radius now that the radiation has cooled to $\approx 2.7K$? You may estimate the answer to within a factor of 2 or so.

(e) (5 pts) How much work is performed by the radiation pressure during this time? Express your answer in terms of the initial volume $V_i$, the initial and final temperatures $T_i$ and $T_f$, respectively, and relevant fundamental constants.
**Problem 6: Radiation from a kiln (15 pts)**

An artisan throws a pot and then fires it in a kiln at 1030°C. A small hole with area $A = 1cm^2$ in one wall of the kiln emits thermal radiation from inside. All answers for this problem may be approximated to within a factor of 2 or so.

(a) (5 pts) How much power is being radiated out of the hole?

(b) (5 pts) Immediately after the artisan removes the pot from the kiln, it is still at the same temperature as the inside of the kiln, but it is coated with a highly reflective surface that reflects about 90% of all light that falls on it across the full spectrum. What is the power being radiated from a $1cm^2$ patch of the surface of the pot?

(c) (5 pts) A second kiln nearby is not fully warmed up yet, so that it is at a temperature of 380°C. What is the ratio of the peak frequency $\omega_2$ of the radiation emitted from a small hole in the second kiln over the peak frequency of the radiation leaving the first kiln? (These peak frequencies are the maximum values of the spectral densities as plotted in the frequency domain, which do not necessarily correspond to the peaks of the densities plotted as a function of the wavelength.)
Here is some potentially useful information:

\[
\begin{align*}
  c &= 3 \times 10^8 \text{m/s} \\
  h &= 6.63 \times 10^{-34} \text{Js} \\
  1\text{eV} &= 1.60 \times 10^{-19} \text{J} \\
  k &= 1.38 \times 10^{-23} \text{J/K} \\
  \sigma &= 5.67 \times 10^{-8} \text{W/(m}^2\text{K}^4) \\
  m_p &= 1.67 \times 10^{-27} \text{kg} \\
  m_e &= 9.11 \times 10^{-31} \text{kg} \\
  N_A &= 6.02 \times 10^{23} \\
  g &= 9.8 \text{m/s}^2
\end{align*}
\]