Above the front door of Niels Bohr's cottage was nailed a horseshoe. A visitor who saw it exclaimed: "Being as great a scientist as you are, do you really believe that a horseshoe above the entrance to a home brings good luck?"
"No," answered Bohr, "I certainly do not believe in this superstition. But you know," he added with a smile, "they say that it does bring luck even if you don't believe in it!"

- George Gamow, excerpted from *Thirty Years that Shook Physics.*

If you have any questions, suggestions or corrections to the solutions, don't hesitate to e-mail me at dfk@uclink4.berkeley.edu!

Problem 5 *Geonium: Motion of an electron in a Penning Trap*

The motion of an electron in a Penning trap (geonium) can be deduced from considering the potential energy of the e^- due to the applied quadrupole electrostatic field, whose potential is given by

$$\phi_E = A\big(x^2 + y^2 - 2z^2\big),$$

the strong, homogeneous magnetic field $\vec{B} = B_0\hat{z}$, and the weak "bottle" magnetic field

$$\vec{b} = -\beta\left(zx\hat{x} - zy\hat{y} + \left(z^2 - \frac{x^2+y^2}{2}\right)\hat{z}\right).$$

(a)

Now we consider the axial oscillation frequency on-axis ($x = y = 0$). In this case, only the electrostatic field and the bottle field contribute to the potential energy. The Hamiltonian H is given by:

$$H = \frac{p_z^2}{2m_e} - \vec{\mu}\cdot\vec{b} - e\phi_E = \frac{p_z^2}{2m_e} + \mu_{\rm eff}\beta z^2 + 2eAz^2, \tag{1}$$

where $\mu_{\rm eff}$ is the effective magnetic moment of the electron due to its spin, cyclotron and magnetron motion. Equation (1) is the Hamiltonian for a simple harmonic oscillator:

$$H = \frac{p_z^2}{2m_e} + \frac{1}{2}m_e\omega_A^2 z^2, \tag{2}$$

where the axial oscillation frequency is given by:

$$\omega_A^2 = \frac{2}{m_e}(\mu_{\rm eff}\beta + 2eA). \tag{3}$$

The second term in Eq. (3) is the unperturbed axial oscillation frequency (i.e. the axial frequency without the bottle field), given by Eq. (2) from H. Dehmelt, Am. J. Phys. **58**, 17 (1990):

$$\omega_z^2 = \frac{4eA}{m_e}. \tag{4}$$

So the axial oscillation frequency in the presence of the bottle field is given by:

$$\omega_A = \left(\frac{2\mu_{\rm eff}\beta}{m_e} + \omega_z^2\right)^{1/2} \approx \omega_z + \frac{\mu_{\rm eff}\beta}{\omega_z m_e}. \tag{5}$$

The correction to the axial oscillation frequency is thus given by:

$$\delta\omega_z \approx \frac{\mu_{\rm eff}\beta}{\omega_z m_e}. \tag{6}$$

Now we need to determine $\mu_{\rm eff}$, which is given by:

$$\mu_{\rm eff} = \mu_s + \mu_c + \mu_m, \tag{7}$$

where μ_s is the spin magnetic moment, μ_c is the cyclotron magnetic moment, and μ_m is the magetic moment due to magnetron motion. The spin magnetic moment is:

$$\mu_s = g_e\mu_B m \approx 2\mu_B m, \tag{8}$$

where $m = \pm 1/2$ is the spin projection along the z-axis and μ_B is the Bohr magneton:

$$\mu_B = \frac{e\hbar}{2m_e c}. \tag{9}$$

The cyclotron motion in the strong, homogeneous magnetic field $\vec{B}$ is described approximately (here we ignore the shift due to the electric quadrupole field) by the balance of the magnetic and centrifugal forces on the electron:

$$\frac{er\omega_c}{c}B_0 = m_e r\omega_c^2, \tag{10}$$

which gives the cyclotron frequency:

$$\omega_c = \frac{e}{m_e c}B_0. \tag{11}$$

This system can also be described with the equations for a simple harmonic oscillator with only a single degree of freedom. Here is a simple derivation of this

fact from L.D. Landau and E.M. Lifschitz, *Quantum Mechanics*, (Butterworth-Heinemann, Oxford, 1977), based on the original derivation by Landau himself in 1930 (the energy levels for this system are known as the Landau levels). The vector potential $\vec{A}$ for a uniform field in the $\hat{z}$ direction can be written, for example, as:

$$\vec{A} = -B_0 y \hat{x}.$$

The Hamiltonian H_c governing this cyclotron motion is then:

$$H_c = \frac{1}{2m_e}\big((p_x + eB_0 y/c)^2 + p_y^2 + p_z^2\big),$$

where we recall that the kinetic momentum $m\vec{v}$ in the presence of a magnetic vector potential is related to the canonical momentum $\vec{p}$ by:

$$m\vec{v} = \vec{p} - \frac{q}{c}\vec{A},$$

where q is the charge and $\vec{A}$ is the vector potential. Note that the Hamiltonian does not contain the coordinates x and z, so $[H_c, p_x] = [H_c, p_z] = 0$, so the eigenvalues of p_x and p_z can take on any real value. Clearly motion in the axial direction ($\hat{z}$) can therefore be decoupled from the cyclotron motion in this case. Therfore the effective Hamiltonian $H_c^{(xy)}$ describing motion in the xy-plane is:

$$H_c^{(xy)} = \frac{1}{2m_e}p_y^2 + \frac{1}{2}m_e\omega_c^2(y - y_0)^2,$$

where $y_0 = -cp_x/(eB_0)$. Therefore the cyclotron motion can be described with the formalism developed for the 1D simple harmonic oscillator, even though it is a two dimensional problem! Furthermore, if we assign an energy $\mu_c B_0$ to each of the Landau levels, we can associate the cyclotron frequency with a magnetic moment according to:

$$\hbar\omega_c\left(n + \frac{1}{2}\right) = \frac{e\hbar}{m_e c}\left(n + \frac{1}{2}\right)B_0 = \mu_c B_0, \tag{12}$$

where n is the cyclotron quantum number. From Eq. (12) we deduce:

$$\mu_c = 2\mu_B\left(n + \frac{1}{2}\right). \tag{13}$$

The magnetron motion arises because the static electric quadrupole field also creates a radial electric force. The magnetron motion can also be solved for using the similar formalism as applied above (see, e.g., L.S. Brown and G. Gabrielse, Rev. Mod. Phys. **58**(1), 233, (1986)). It turns out that the magnetic moment due to the magetron motion is simply the ratio of the magnetron and cyclotron frequencies times the magnetic moment due to the cyclotron motion (with a new quantum number q to describe the magnetron motion):

$$\mu_m = 2\mu_B\left(\frac{\omega_m}{\omega_c}\right)\left(q + \frac{1}{2}\right). \tag{14}$$

Employing the above relations in Eq. (6), we find:

$$\boxed{\delta\omega_z \approx \frac{2\mu_B\beta}{\omega_z m_e}\left(m + n + \frac{1}{2} + (q + 1/2)\frac{\omega_m}{\omega_c}\right)}$$

agreeing with the equation from Dehmelt's paper, if we neglect the 1/2 that goes with the magnetron motion (since under typical experimental conditions $q \gg 1$).

(b)

We cannot in fact require that $m = 0$, since the spin quantum number can only take on the values $m = \pm 1/2$. Therefore, let's choose the minimum possible perturbation of the axial oscillation frequency $\delta\omega_z^{(min)}$, which occurs when $m = -1/2$, $n = 0$, and $q = 0$. It might also be argued that $q = 0$ is an unreasonable assumption, since the magnetron frequency is 13 kHz and at the working temperature for geonium, $kT \approx 100$ GHz. Putting these concerns aside for the moment, the minimum perturbation to the axial oscillation frequency is given by:

$$\delta\omega_z^{(min)} = \frac{1}{2}\frac{\omega_m}{\omega_c}\frac{2\mu_B\beta}{\omega_z m_e}. \tag{15}$$

This leads to the overall axial oscillation frequency:

$$\omega_A = \Big(\omega_z + \delta\omega_z^{(min)}\Big)\left(a + \frac{1}{2}\right), \tag{16}$$

where ω_z is the unperturbed axial oscillation frequency from Eq. (4) and a is the quantum number for the axial oscillation. We choose the ground state of such a system ($a = 0$), which is described by the wavefunction:

$$\psi_g(z) = \left(\frac{m_e\omega_0}{\hbar\pi}\right)^{1/4} e^{-m_e\omega_0 z^2/2\hbar}, \tag{17}$$

where $\omega_0 = \frac{1}{2}\Big(\omega_z + \delta\omega_z^{(min)}\Big)$. Since

$$\omega_z \approx 2\pi \cdot 60 \text{ MHz}$$

and

$$\delta\omega_z^{(min)} \approx \frac{2\pi \cdot 13 \times 10^3 \text{ Hz}}{2\pi \cdot 141 \times 10^9 \text{ Hz}}(2\pi \cdot 1 \text{ Hz}),$$

we have $\omega_0 \approx \omega_z$.
From symmetry considerations $\langle z \rangle = 0$, and a straightforward integral gives:

$$\langle z^2 \rangle = \int_{-\infty}^{\infty} dz \Big(\frac{m_e\omega_z}{\hbar\pi}\Big)^{1/2} z^2 e^{-m_e\omega_z z^2/\hbar} = \frac{\hbar}{2m_e\omega_z}. \tag{18}$$

Thus the dimensions of the wavepacket are given by:

$$(\Delta z)^2 = \langle z^2 \rangle - \langle z \rangle^2 = \frac{\hbar}{2m_e\omega_z}. \tag{19}$$

Thus the minimum dimension of the electron wavepacket is:

$$\boxed{\Delta z = \sqrt{\frac{\hbar}{2m_e\omega_z}} \approx 3 \times 10^{-5}\text{cm}}$$

Since the wavepacket may have different radial and axial dimensions, we should also consider the radial motion of the electron. This motion can also be treated as a simple harmonic oscillator, so we find that:

$$(\Delta r)^2 = \frac{\hbar}{2m_e\omega_m}, \tag{20}$$

where $\omega_m \approx 2\pi \cdot 13$ kHz is the magnetron frequency. This gives us a radial dimension of the electron wavepacket:

$$\boxed{\Delta r = \sqrt{\frac{\hbar}{2m_e\omega_m}} \approx 3 \times 10^{-3}\text{cm}}$$

Problem 6 *Zeeman Slower*

(a)

We assume that the atoms are always in resonance, and the optical pumping saturation parameter $\kappa = 1$. The relevant optical pumping saturation parameter in this case is given by:

$$\kappa = \frac{d^2E^2}{\gamma_0^2}, \tag{21}$$

where d is the transition dipole moment, E is the electric field of the light, and γ_0 is the natural width of the excited state. Essentially, this says that the rate of pumping atoms from the ground state to the excited state,

$$\Gamma_p \approx \frac{d^2E^2}{\gamma_0},$$

is equal to the rate of spontaneous decay from the excited state γ_0.

The average force $\langle F \rangle$ transferred to the atoms from the laser light is given by:

$$\langle F \rangle = \frac{\Delta p}{\Delta t} = \frac{\hbar k}{4\tau}, \tag{22}$$

where $\hbar k$ is the momentum of an absorbed photon and τ is the lifetime of the upper state. There is no average momentum imparted to the atoms from spontanteous emission since photons are spontaneously emitted in (approximately) random directions. Δt is the time for one cycle of pumping and spontaneous emission. Since $\kappa = 1$, the rates of spontaneous emission, stimulated emission, and stimulated absorption are the same ($\approx \gamma_0$). We can then estimate that each cycle of absorption and emission takes 2τ, and roughly half the time such a cycle involves spontaneous emission (no momentum is imparted to the atom by stimulated absorption followed by stimulated emission). Thus the effective time for slowing an atom by $\hbar k/M$ is 4τ. The average force for $\lambda = 589$ nm and $\tau = 16$ ns is

$$\langle F \rangle \approx 2 \cdot 10^{-15} \text{ g} \cdot \text{cm} \cdot \text{s}^{-2}.$$

The initial thermal velocity of the atoms effusing from the oven (assuming a Maxwell distribution) is given by (see, e.g., F. Reif, *Fundamentals of Statistical and Thermal Physics*, (McGraw-Hill, New York, 1965), pp. 268-9):

$$\langle v \rangle = \sqrt{\frac{8kT}{\pi M}}, \tag{23}$$

where M is the mass of sodium. From this we find the average initial momentum $\langle p_0 \rangle$ of a sodium atom is:

$$\langle p_0 \rangle = \sqrt{\frac{8kTM}{\pi}} \approx 3 \cdot 10^{-18} \text{ g} \cdot \text{cm} \cdot \text{s}^{-1}, \tag{24}$$

which corresponds to a thermal velocity of $9 \cdot 10^4$ cm/s. The stopping time is given by the ratio of the average light force to the initial momentum,

$$\boxed{t_{\text{stop}} \approx 1.5 \times 10^{-3} \text{ s}}$$

In the experiment, $t_{\rm stop} \approx 3 \cdot 10^{-3}$ s, so $\kappa_{\rm exp} \sim 1/2$.

(b)

In this part of the problem, we assume all atoms have the same initial velocity and neglect hyperfine structure, and assume $\kappa = 1$. The atoms are always kept in resonance by the magnetic field, so the atoms undergo uniform deceleration:

$$a = \frac{\langle F \rangle}{M} = \frac{\hbar k}{4\tau M} \approx 5 \times 10^7 \text{ cm} \cdot \text{s}^{-2}. \tag{25}$$

The light frequency is Doppler shifted by an amount

$$\Delta\omega(z) = \omega \frac{v(z)}{c}, \tag{26}$$

where z is the distance from the trap and $v(z)$ is the atomic velocity as a function of the distance. The velocity as a function of distance in terms of the deceleration a is

$$v(z) = \sqrt{2az}.$$

In order to keep the atoms on resonance, we need to equate the Zeeman shift created by the magnetic field to the Doppler shift $\Delta\omega(z)$.

For σ^+ polarized light, atoms tend to be pumped into the $M_J = 1/2$ ground state Zeeman sublevel. Since atoms experience many optical pumping cycles, it is the energy difference between the $3\ ^2S_{1/2}\ M_J = 1/2$ and $3\ ^2P_{3/2}\ M_J = 3/2$ states which is important for slowing the atomic beam. The Zeeman shift of the resonance frequency for σ^+ polarized light is given by

$$\Delta\omega = (g_e \cdot (3/2) - g_g \cdot (1/2)) \frac{\mu_B B(z)}{\hbar}, \tag{27}$$

where $g_e = 4/3$ and $g_g = 2$ are the excited and ground state Landé factors (see HW 1 solutions Eqn. (29)), μ_B is the Bohr magneton and $B(z)$ is the magnetic field as a function of distance from the trap. By equating Eqns. (26) and (27), we find:

$$\boxed{B(z) = \frac{\hbar\omega_0\sqrt{2az}}{\mu_B c}}$$

Numerically, this result is:

$$B(z) = \left(\frac{2\pi \times 5 \cdot 10^{14} \text{ Hz}}{2\pi \times 1.4 \cdot 10^6 \text{ Hz/G}} \frac{1}{3 \cdot 10^{10} \text{ cm/s}} \sqrt{10^8 \text{ cm} \cdot \text{s}^{-2}} \right) \sqrt{z} \approx 1.2 \cdot 10^2 \sqrt{z} \text{ G}.$$

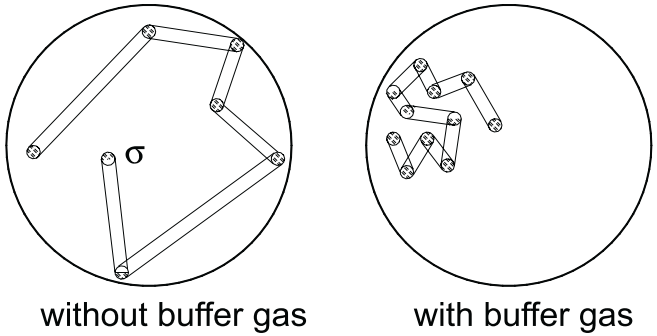

Figure 1: Molecule moving about a cell with and without buffer gas. Although the total collision frequency in the cell with buffer gas is higher, the effective volume traced out by the molecule in a given time t is unchanged.

Problem 7 *Collisions in gas mixture*

The mean free path λ of a molecule between collisions is given by:

$$\lambda = \frac{1}{n\sigma}, \tag{28}$$

where n is the density and σ is the collisional cross section. The average time between collisions is $\lambda/\bar{v}$, where $\bar{v}$ is the average relative thermal velocity.

If a second buffer gas with density n' and cross section σ' is added into the system, the total collision rate γ is:

$$\gamma = n\sigma\bar{v} + n'\sigma'\bar{v}' = \gamma_{\rm self} + \gamma_{\rm buffer}, \tag{29}$$

where $\bar{v}'$ is the average relative velocity between the molecules and buffer gas, and $\gamma_{\rm self}$ and $\gamma_{\rm buffer}$ are the rates for self- and buffer gas collisions, respectively. Note that the collision rate between the original molecules is in fact *unmodified*, there are just more total collisions! Thus the introduction of the buffer gas doesn't change the time between collisions of the original molecules with themselves.

This result is illustrated in Fig. 1: a molecule moving about the cell traces out an effective volume $\sigma\bar{v}t$, and the probability of a collision becomes unity about when the volume traced out $= 1/n$. These parameters are independent of the frequency of collisions with the buffer gas. Note that this result is only true for equilibrium conditions.