Relation between the Josephson frequency and the Arrhenius rate for the nucleation of 2π phase slips in superfluid ^{4}He

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The creation rate of phase slip vortices is related to an energy barrier through an Arrhenius law. By computing a proportionality factor that relates the Josephson frequency to the Arrhenius rate one can then deduce the form of the vortex nucleation energy barrier from the observed dependence of critical velocity on pressure drive. The results of a numerical simulation is presented which evaluates this factor over a large parameter space.

When superfluid ^{4}He flows through a submicrometer-sized aperture, energy can be dissipated by 2π phase slip events.[1,2] These discrete energy loss processes occur at a well defined critical velocity v_c, at which a quantized vortex is nucleated near the surface of the aperture and subsequently grows in the ambient flow. Within a few microseconds the vortex element crosses all the flow lines through the aperture, before eventually annihilating on a distant boundary.[3] The energy carried away[4] by the vortex causes the velocity in the aperture to drop by $\delta v = \kappa / l$ where κ is the circulation quantum (the ratio of Planck's constant to the ^{4}He atomic mass) and l is the effective hydraulic length of the aperture.

If the flow through the aperture is sustained by an impressed pressure head Δp, the spatially averaged flow velocity in the aperture consists of a continuous sawtooth wave form in which a linear acceleration is followed by an abrupt 2π deceleration event. Equating the force on the fluid to the product of mass and acceleration leads to the frequency of the sawtooth[4]

$$f_J = \frac{\Delta p}{\rho \kappa} . \tag{1}$$

Here ρ is the liquid density and $\Delta p / \rho$ is the chemical potential difference which is the fundamental driving potential for superfluid flow. The Josephson frequency f_J is so-named because of the similarity of Eq. (1) with the corresponding ac Josephson equation in superconductors. [However, for the case of ^{4}He, Eq. (1) has nothing to do with the dc Josephson relationship.]

The quantized vortices which cause the phase slips are created by thermal-activation processes which carry the system over a velocity-dependent energy barrier $E(v,T)$. The height of this barrier is a decreasing function of velocity and is a generic consequence of the nearby boundary. The barrier prevents nascent vortices from entering the fluid until either the temperature is high enough to overcome the barrier or the velocity is high enough to lower the barrier.[5] Since each vortex nucleation event causes a phase slip, the rate of phase slips is given by the Arrhenius law

$$f_A = \Gamma e^{-[E(v,T)/k_B T]} = e^{-[E^*(v,T)/k_B T]} . \tag{2}$$

Here Γ is an attempt frequency for the process and $E^* \equiv E(v) - k_B T \ln \Gamma$.

Recent attention[6,7] has focused on the problem of determining the functional form of $E^*(v,T)$. Since a phase slip decreases the velocity, it is difficult to directly probe $E^*(v,T)$ through the Arrhenius formula, which describes a rate at constant velocity. One experimental approach is to study pressure-driven flow through an aperture.[6,7,8] It is possible to measure the dependence of the average critical velocity v_c on the externally applied pressure head Δp, at constant temperature. The inverse function $\Delta p(v_c)$, when inserted into Eq. (1), then relates the Josephson frequency to the critical velocity $f_J(v_c)$. One is then faced with the problem of relating f_J to f_A and thereby to E^*. We will show below that it is possible to write $f_J = \gamma f_A$, where γ is a calculable function of the phase slip size, δv. Then a combination of Eqs. (1) and (2) yields

$$E^* = -k_B T \ln \left[\frac{\Delta p(v_c)}{\gamma \rho \kappa} \right] . \tag{3}$$

The purpose of this work is to find the factor γ for all possible values of the phase slip size.

The thermal fluctuations give rise to a statistical width α, in the distribution of critical velocities. This width can be defined in terms of the velocity derivative of the energy barrier:

$$\alpha \equiv - \frac{k_B T}{[\partial E(v,T)/\partial v]|_{v_c}} . \tag{4}$$

From this definition we see that α is the velocity range in which thermal fluctuations, of size $k_B T$, can assist vortices over the energy barrier.

If the energy barrier decreases linearly with velocity and if the phase slip size $\delta v \gg \alpha$, it has been shown[9] that

$$v_c = v_{co} \left[1 - \frac{k_B T}{E_0} \ln \left[1.78 \frac{k_B T}{E_0} \Gamma v_{co} \frac{\rho l}{\Delta p} \right] \right] . \tag{5}$$

The approximation made to derive Eq. (5) was based on the assumption that the phase slip size is sufficiently large so that the nucleation rate drops to essentially zero after a phase slip. This is the case if the phase slip size is large compared to the width of the distribution. Here the linear energy barrier has been taken to be of the form

$$E(v,T)=E_0(1-v/v_{co}) , \quad (6)$$

where $-E_0/v_{co}$ is the slope of $E(v,T)$.

If $\rho\kappa f_J$ is substituted for Δp, one can solve Eq. (5) for f_J:

$$f_J=\gamma\Gamma e^{-[E(v_c,T)/k_BT]}=\gamma f_A(v_c) , \quad (7)$$

where

$$\gamma\equiv 1.78\alpha/\delta v . \quad (8)$$

The factor γ is the required function to relate f_J and f_A. From Eq. (7), γ can be seen as a correction factor for the attempt frequency Γ in the Arrhenius formula. This is reasonable because, for the case of pressure driven flow, the velocity spends only part of its time near v_c whereas $f_A(v_c)$ represents a nucleation rate appropriate for the velocity remaining at v_c.

Since δv depends on the hydraulic length of the aperture, the limit $\alpha/\delta v \ll 1$ does not always apply, so we seek a more general form of the factor γ in terms of the ratio $\alpha/\delta v$. When $\alpha/\delta v$ is large enough that Eq. (8) is not valid, then after a vortex has just crossed the aperture, there may still be a significant possibility that another vortex will be formed. For large $\alpha/\delta v$, which corresponds to a very small decrease in velocity after a phase slip, the velocity remains close to v_c. In this case, the rate of creation of vortices f_J is given by the Arrhenius rate, Eq. (2), evaluated at v_c. Thus, γ is expected to tend to unity for large $\alpha/\delta v$.

A value of γ greater than unity is unphysical. This can be seen by recalling that γ can be taken as a correction factor to the attempt frequency. The correction is required because, during the sawtooth, v is usually less than v_c. Note that Eq. (8) implies that γ goes to infinity for large $\alpha/\delta v$. If γ were greater than unity, the corrected attempt frequency would be greater than the actual attempt frequency. Thus, Eq. (8) is clearly invalid for large $\alpha/\delta v$.

The parameter α can be determined from the observed dependence of v_c on pressure head, Δp. This is seen from Eq. (7) by writing f_J in terms of Δp [Eq. (1)] and then taking the natural logarithm of both sides. Subsequent differentiation with respect to v_c yields

$$\alpha\approx\frac{dv_c}{d\ln(\Delta p)} . \quad (9)$$

Here we have assumed that γ is independent of v_c, which is equivalent to assuming that $E(v,T)$ is approximately linear near v_c. $E(v,T)$ is approximately linear if $d\alpha/dv_c \ll 1$. Since the dependence of critical velocity on driving pressure can be measured, α can be determined from Eq. (9) using the observable, $v_c(\Delta p)$. Typical values from experiments show α is on the order of a few percent of v_c.[6,7,10]

The value of γ for arbitrary values of $\alpha/\delta v$ can be found by numerical simulation. For pressure driven flow (i.e., fixed f_J), the equation of motion of the superfluid is a combination of uniform acceleration punctuated by discrete velocity decrements. In the simulation the velocity is incrementally advanced in small time steps and the Arrhenius rate f_A is used to determine the probability P of a phase slip nucleation event in the time step Δt.

$$P=1-e^{-f_A\Delta t} . \quad (10)$$

A random number N, less than unity, is selected in each step. If $N<P$ the velocity is decreased by δv (a phase slip event), superposed on the acceleration change. If $N>P$ the velocity is only increased. This simulation creates the sawtooth $v(t)$ which is characteristic of the pressure driven flow. From this sawtooth we compute v_c, the average value of the velocity at which the phase slips occur. This value is then inserted into the Arrhenius formula Eq. (2) to get $f_A(v_c)$. The ratio of f_J to this number gives γ.

The simulation could be performed over a wide range of the parameters f_J and δv, but this can be avoided by choosing appropriate units. We are free to choose the units of both velocity and time independently. We choose to measure velocity in units of δv. With this choice, the velocity is decreased by 1 after a vortex is created. Furthermore, we choose to measure time in units of f_J^{-1}. Therefore, on the average, a vortex crosses the aperture every 1 time unit.

In these units, the equation of motion of the fluid is given by

$$dv/dt=1 . \quad (11)$$

This equation of motion is the statement that a constant acceleration is imposed on the fluid in the aperture.

For the linearized energy barrier, the rate of vortex crossings given by Eq. (2) is of the form:

$$f_A=be^{v/\alpha} . \quad (12)$$

We hold the temperature fixed in the simulation, and there is no need to show the temperature dependence of the parameter b. During each time step, the probability of a vortex being created is given by Eq. (10), where f_A is given by Eq. (12).

The simulation yields v_c, the average velocity at which a vortex is created. The rate f_A of vortex creation at fixed velocity, given in Eq. (12), is then evaluated at this v_c. The desired factor γ is the ratio of f_J (unity in our units) to f_A. Specifically,

$$\gamma=(be^{v_c/\alpha})^{-1} . \quad (13)$$

By Eq. (12), the parameter b is the rate of vortex creation at zero velocity. If this parameter is very small, a vortex will have a negligible probability of being created, except when the velocity is large. Therefore, small b yields a large value of v_c. Similarly, large b corresponds to small v_c. By Eq. (8), we expect $\gamma=1.78\alpha$ for small α (δv is unity in the simulation). Therefore, we expect that γ is independent of the choice of b, for small α. This is in fact the obtained result. Furthermore, it is found that the values of γ do not depend on b for any α. We must

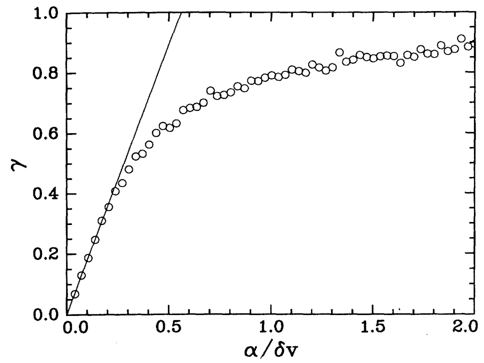

FIG. 1. The values of the correction factor γ obtained from simulation. The solid line is the expression in (8).

choose some reasonable value of b, even though the general form of γ does not depend on b. Since f_J and $f_A(v_c)$ should be on the same order of magnitude, we can bracket the range of possibilities of b. Our simulation has been performed for values of $b=e^{-50}$, e^{-100}, and e^{-200}. These correspond to $v_c/\alpha \approx 50$, 100, and 200. The final values of γ are found not to depend on b in this range. In summary, we choose b and α, and the simulation gives v_c. The quantity γ is evaluated by Eq. (13).

The values of γ obtained from the simulation are shown as open circles in Fig. 1. They are plotted as a function of $\alpha/\delta v$ (which is just α in the units of the simulation). The simulation was carried out for values of $\alpha/\delta v$ up to 10. The values of γ do not depend on b for any α.

In Fig. 1, γ is seen to approach the anticipated value of unity for large $\alpha/\delta v$, reflecting the fact that if the phase slip velocity decrement δv is very small, the velocity remains near v_c always, and $f_J \approx f_A$. Conversely for small values of $\alpha/\delta v$, γ is seen to equal the value $1.78\alpha/\delta v$, as given by Eq. (7).

TABLE I. The coefficients for the numerical fit to γ. The fit was made in two temperature regimes, as indicated. γ is given by $\sum_{n=0}^{6} a_n[\ln(\alpha/\delta\nu)]^n$.

	$0.0487 < \alpha/\delta\nu < 1.432$	$1.432 < \alpha/\delta\nu < 10$
a_0	0.7849646	0.7783844
a_1	0.1731825	0.1943483
a_2	-6.860631×10^{-2}	-7.229038×10^{-2}
a_3	3.104779×10^{-2}	-5.644377×10^{-3}
a_4	2.973394×10^{-2}	7.406305×10^{-3}
a_5	4.774672×10^{-3}	4.127597×10^{-4}
a_6	4.062508×10^{-5}	-1.705651×10^{-4}

The behavior in the range $0.05 < \alpha/\delta v < 10$ can be expressed as a polynomial

$$\gamma = \sum_{n=0}^{6} a_n[\ln(\alpha/\delta\nu)]^n , \tag{14}$$

where the coefficients a_n are given in Table I.

In conclusion, we have found a proportionality function γ which connects the Josephson frequency f_J to the Arrhenius rate f_A. This creates a direct link, by Eq. (3), between an experimentally measurable function $\Delta\rho(v_c)$ and the nucleation energy function $E^*(v)$. If $\delta v \gg \alpha$ then Eq. (5) holds. If $\delta v \ll \alpha$ then the velocity stays close to v_c and γ tends to one. When δv is of the order of α our numerical simulation provides the value of γ.

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