

Velocity-dependent effective inertial mass in superfluid ^3He

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We show that a solid body moving in superfluid ^3He acquires a velocity-dependent effective mass. The effect is a consequence of the suppression of the superfluid density associated with depairing in a flowing BCS fluid. The counterflow associated with the body's motion is greater than for a simple inviscid fluid. To leading order, the mass increase is quadratic in velocity. We show that this effect leads to nonlinearities in the motion of a simple system such as a solid object subject to a linear restoring force. Our model may explain some of the nonlinear behavior seen in vibrating wire oscillators at very low temperatures. The depairing term will also contribute to the low-velocity counterflow mass increase in excess to what is expected from the ideal fluid model.

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The many correspondences between phenomena in superfluid ^3He and the general paradigm of modern physics have been emphasized by Volovik.¹ Here we describe how the relative motion of a superfluid changes its inertia, analogous to the relativistic mass increase. The question we address is, "What is the inertial mass of a probe particle moving in the superfluid?" When a solid object moves with a velocity u through an inviscid background fluid, the potential flow in the fluid couples extra inertia to the object.² The extra effective mass is proportional to the fluid's density. In a BCS superfluid such as ^3He , the superfluid density ρ_s is a decreasing function of the superfluid velocity, defined as the relative velocity between the rest frame of the superfluid and the background normal fluid: $\rho_s = \rho_s(v)$. This decrease in ρ_s is accompanied by the creation of quasiparticles, equivalent to enhancing the normal component in the two-fluid description. A regime exists where the additional normal component, $\delta\rho_n$, is clamped to the moving body, also contributing to the body's inertia. These contributions, along with a dynamic contribution arising from the Cooper depairing, add a term δm to the object's static mass. Thus, the velocity-induced depairing phenomenon will render the term δm velocity dependent. This paper focuses on estimating the magnitude of the velocity dependent corrections to the inertial mass of a moving particle within the superfluid. We show that, to leading order, the effective mass of the moving object $m(u) \approx m_{u=0}^* (1 + u^2/2c^{*2})$, where u is the object's velocity, and the temperature-dependent velocity scale c^* is determined by the BCS energy gap and the Fermi fluid parameters.

We will discuss the case of low temperatures (i.e., $T \ll T_c$), where the background density of quasiparticles is low. At the same time, we consider the motion of large objects, so that the quasiparticle mean free path is small in comparison with the object size. In this regime, the quasiparticles (which are related to the suppression of ρ_s) are created in the frame of the moving body, and are reabsorbed into the condensate when the body decelerates. We assume that there is no associated irreversible energy loss. At higher tempera-

tures, the background density of the quasiparticles becomes comparable to that produced by the Cooper depairing, and a more detailed analysis of the effect would be required.

If the kinetic energy of an object has velocity terms other than quadratic, the object's effective mass, $m(u) \equiv p(u)/u$, is velocity dependent. For an object moving through superfluid $^3\text{He-B}$, this energy can be broken into four components: the kinetic energy of the solid body itself, E_m , the kinetic energy of the superfluid counterflow $E_{k,s}$, the kinetic energy of the quasiparticles, created via velocity suppression of ρ_s , $E_{k,n}$, and a term E_C , associated with the breaking of the Cooper pairs. The last three terms can be calculated by integrating position - dependent energy densities.

At low temperatures and small relative velocities we choose to parameterize the superfluid density of ^3He by a Taylor expansion (see Fig. 1):

$$\rho_s(v) = \rho_0 \left[1 - A \left(\frac{v}{v_c} \right)^2 - B \left(\frac{v}{v_c} \right)^4 + \dots \right], \quad (1)$$

where ρ_0 is the superfluid density at rest. The normal component density of helium at rest is negligible for $T/T_c \ll 1$. The characteristic depairing velocity v_c is the ratio of the superfluid energy gap Δ , to the Fermi momentum p_F , $v_c \equiv p_F^{-1} \Delta$, and determines the velocity scale. The temperature dependent, dimensionless coefficients A and B can be determined from, for example, the theoretical calculation by Vollhardt *et al.*³

The kinetic energy term due to the superfluid counterflow (in the laboratory frame) can be calculated by integrating the energy density in the potential velocity flow field induced by the moving object

$$\begin{aligned} E_{k,s} &= \frac{1}{2} \int_{Bulk} \rho_s(\mathbf{r}) v^2(\mathbf{r}) d^3\mathbf{r} \\ &= \frac{1}{2} \rho_0 \int_{Bulk} v^2(\mathbf{r}) \left[1 - A \left(\frac{v(\mathbf{r})}{v_c} \right)^2 - B \left(\frac{v(\mathbf{r})}{v_c} \right)^4 \right] d^3\mathbf{r}. \end{aligned} \quad (2)$$

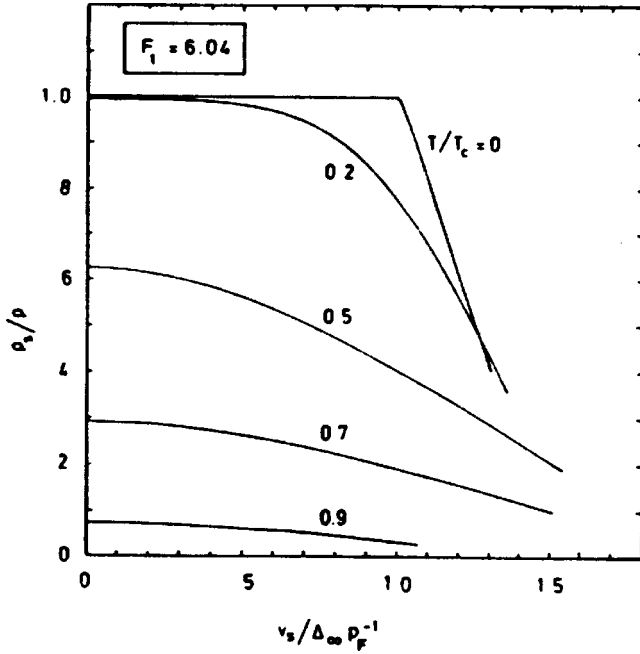


FIG. 1. Superfluid density of ^3He as a function of the fluid velocity at zero pressure, courtesy of a theoretical calculation by Vollhardt *et al.* (Ref. 3).

The next contribution to the kinetic energy is due to the quasi-particles created in the frame of the moving solid body. The number density of these is position dependent, but all are assumed to move with the velocity u of the solid body. The kinetic energy associated with these quasi-particles is expressed as the integral

$$E_{k,n} = \frac{1}{2} u^2 \int [\rho_0 - \rho_s(\mathbf{r})] d^3\mathbf{r} \\ = \frac{1}{2} \rho_0 u^2 \int_{Bulk} \left[A \left(\frac{v(\mathbf{r})}{v_c} \right)^2 + B \left(\frac{v(\mathbf{r})}{v_c} \right)^4 \right] d^3\mathbf{r}. \quad (3)$$

The depairing term is the energy needed to break the Cooper pairs. It is proportional to the number density of the created quasiparticles and the depairing energy Δ per pair:

$$E_C = \frac{\Delta}{2m_3} \int (\rho_0 - \rho_s)(\mathbf{r}) d^3\mathbf{r} \\ = \frac{\rho_0}{2m_3} \Delta \int \left[A \left(\frac{v(\mathbf{r})}{v_c} \right)^2 + B \left(\frac{v(\mathbf{r})}{v_c} \right)^4 \right] d^3\mathbf{r}. \quad (4)$$

The total kinetic (velocity-dependent) energy of the moving object is the sum of these three terms, plus the kinetic energy of the object itself. We will keep terms up to quartic in the solid body velocity. Quadratic contributions are from the object itself, the hydrodynamic counter-flow and Cooper depairing:

$$T_2 = \frac{1}{2} M u^2 + \frac{1}{2} \rho_0 \int_{Bulk} v^2(\mathbf{r}) \left(1 + \frac{A\Delta}{m_3 v_c^2} \right) d^3\mathbf{r}. \quad (5)$$

This term will result in overall inertial mass increase compared to vacuum.

The fourth order terms are

$$T_4 = \frac{\rho_0}{2} \int_{Bulk} v^2(\mathbf{r}) \left[A \left(\frac{u^2}{v_c} \right)^2 - A \left(\frac{v(\mathbf{r})}{v_c} \right)^2 \right. \\ \left. + \frac{B\Delta}{m_3 v_c^2} \left(\frac{v(\mathbf{r})}{v_c} \right)^2 \right] d\mathbf{r}. \quad (6)$$

The flow velocity field in the superfluid is potential, even though the superfluid density varies as a function of position. For a given flow geometry, one can in principle calculate the flow velocity field consistent with the boundary conditions imposed by the object's shape, and then evaluate the various contributions described above. This will remain the case until the flow velocity reaches the depairing critical velocity at any point near the object surface. In this case, a self-consistent solution with variable boundary conditions would need to be found. However, our calculations here are based on the Taylor expansion estimates of the depairing effects and are not expected to hold at these higher velocities. Calculations for a solid sphere and for a circular cross-section wire can be performed in closed form, and each of Eqs. (2), (3), and (4) can be expressed in terms of the velocity of the object.⁴

For a circular cross-section cylinder (such as the case of a section of a wire viscometer) the quadratic and the quartic terms in the effective kinetic energy are

$$T_2 = \frac{1}{2} M u^2 \left[1 + \frac{\rho_0}{\rho_{Cylinder}} \left(1 + \frac{A\Delta}{m_3 v_c^2} \right) \right] \quad (7)$$

$$T_4 = \frac{\rho_0}{2\rho_{Cylinder}} \frac{M u^4}{v_c^2} \left(\frac{2}{3} A + \frac{B\Delta}{m_3 v_c^2} \right). \quad (8)$$

The same calculations, performed for a solid sphere, give

$$T_2 = \frac{1}{2} M u^2 \left[1 + \frac{\rho_0}{2\rho_{Sphere}} \left(1 + \frac{A\Delta}{m_3 v_c^2} \right) \right] \quad (9)$$

and

$$T_4 = \frac{\rho_0}{2\rho_{Sphere}} \frac{M u^4}{v_c^2} \left(\frac{2}{5} A + \frac{B\Delta}{2m_3 v_c^2} \right). \quad (10)$$

In both cases, the kinetic energy term can be expressed as a sum of the quadratic and quartic terms in the object's velocity. The kinetic energy for the body immersed in ^3He can be written as

$$T = \frac{1}{2} M u^2 \left(1 + \frac{3}{4} \frac{u^2}{c^{*2}} \right), \quad (11)$$

where the velocity scale is

$$c_{Cylinder}^{*2} = \frac{3v_c^2 \rho_{Cylinder}}{4\rho_0} \left(\frac{2}{3}A + \frac{B\Delta}{m_3 v_c^2} \right)^{-1},$$

$$c_{Sphere}^{*2} = \frac{3v_c^2 \rho_{Sphere}}{4\rho_0} \left(\frac{2}{5}A + \frac{B\Delta}{2m_3 v_c^2} \right)^{-1}. \quad (12)$$

The hydrodynamic correction to the “rest” mass is neglected in the above formulas.

Equation (11) implies that an oscillator formed by a body immersed in superfluid ^3He and subject to a linear restoring force, $F = -kx$, is a nonlinear oscillator, equivalent to a semirelativistic oscillator in the small amplitude regime.⁵ The frequency of such an oscillator is a function of the oscillator amplitude.⁶ At higher amplitudes the effective mass is larger and the resonant frequency is reduced. The resonance curve in the response vs. frequency dependence measurements would display deviation from Lorentzian shape, and if the dissipative term is small, such a system will exhibit a hysteretic behavior.

The “rest mass” correction, in the case of a cylinder,

$$\frac{\delta m}{m} = \frac{\rho_0}{\rho_{Cylinder}} \left(1 + \frac{A(T)\Delta(T)}{m_3 v_c^2} \right), \quad (13)$$

will shift the central frequency of the resonance lower even for low amplitudes. The first contribution is the well-known hydrodynamic backflow term which should shift the frequency by $\delta f/f \approx \rho_0/\rho_{Cyl} \approx 0.5\%$ for a Tantalum wire and ^3He at zero pressure. The second contribution appears only because the superfluid density suppression has a quadratic leading term, which is substantial in a BCS superfluid.¹⁵ The magnitude of the correction is temperature dependent, as both the superfluid energy gap Δ and the coefficient A are temperature dependent. At $T = 0.5T_c$, the magnitude of the second contribution is $\approx 0.12\%$. In wire viscometer measurements (e.g., Ref. 7), this effect appears to be masked by the inertial contribution from the viscously clamped normal component. The two contributions have different tempera-

tures and object sizes vs. viscous length dependencies and may be distinguishable in a detailed study.

A manifestation of the velocity-dependent mass will appear in an oscillator comprising the probe particle coupled to a linear restoring force. Such a system is a nonlinear oscillator characterized by a non-Lorentzian resonant line shape and a hysteretic response. Such features have been identified for thin vibrating wires moving in ^3He at very low temperatures. Since it is known that such vibrating wires have intrinsic nonlinearities themselves, it is of interest to ask how much, if any, of the observed nonlinearities originates within the superfluid itself. A less ambiguous measurement could be made using an oscillating magnetically suspended microsphere.⁸

Nonlinear, hysteretic behavior had been observed in wire viscometer measurements by Guénault *et al.*⁹ at 0.14 mK. Frequency shifts as large as 20 mHz ($\delta f/f \approx \times 10^{-5}$) were observed. König *et al.*¹⁰ attributed much of the effect to a combination of wire heating artifacts and the low-energy excitation levels present in the wire metal. The data provided in Ref. 9 allow us to estimate the wire velocity peak amplitude (≈ 1 cm/s at $\approx 50\text{-}\mu\text{A}$ drive). The fractional frequency shift, evaluated using Ref. 6 with 0.2-mK values of $A = 0.023$ and $B = 0.026$ from Ref. 3, is $\approx 6 \times 10^{-6}$. Although the wire diameter in Ref. 9 was only $\approx 20\%$ of the estimated quasiparticle mean free path, the proposed effect is of the same order as the observed effect.

In conclusion, we propose a model in which the hydrodynamic back-flow contribution to the mass of a solid object moving in a BCS superfluid acquires nonlinear, velocity-dependent terms. This effect should result in amplitude-dependent and hysteretic resonance response curves. The magnitude of the predicted effect is consistent with observed low-temperature behavior of vibrating wire viscometers.

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¹G. Volovik, *Exotic Properties of Superfluid Helium 3* (World Scientific, Singapore, 1992).

²L.D. Landau and E.M. Lifshitz, *Fluid Mechanics* (Pergamon Press, Oxford, 1989).

³D. Vollhardt, K. Maki, and N. Schopohl, *J. Low Temp. Phys.* **39**, 79 (1980).

⁴For flow in two dimensions, as in the case of an infinite wire of radius 1, the velocity potential is $\phi(r, \theta) = -1/r \cos \theta$. The flow velocity is the gradient of this potential. We can evaluate the integrals over the space outside the solid body corresponding to the 2nd and the 4th powers of the velocity: $\int_{r>1} \gamma v^2(r, \theta) = \gamma u^2 \int_1^\infty 2\pi r (1/r^4) dr = \gamma \pi u^2$, $\int_{r>1} \epsilon v^4(r, \theta) = \epsilon u^4 \int_1^\infty 2\pi r (1/r^8) dr = \frac{1}{3} \epsilon \pi u^2$. These compare with $\gamma \pi u^2$ and $\epsilon \pi u^4$ for the volume displaced by the solid body. In three dimensions, $\phi(r, \theta) = -\cos \theta / 2r^2$. The relevant integrals are $\int_{r>1} \gamma v^2(r, \theta) = 2\gamma \pi u^2 / 3$ and $\int_{r>1} \epsilon v^4(r, \theta) = 2\epsilon \pi u^4 / 15$,

which compares to $4\gamma \pi u^2 / 3$ and $4\epsilon \pi u^4 / 3$ for the energy contribution from the liquid displaced by the sphere.

⁵Consider the lowest order correction to the period of a semirelativistic oscillator. The energy of a relativistic mass, to the lowest order, is $E = U(x) + (mc^2/\sqrt{1-v^2/c^2}) = U(x) + mc^2 + \frac{1}{2}mv^2 + \frac{3}{8}m(v^4/c^2) + \dots$. The frequency of such oscillator, attached to a classical massless spring with potential energy $U(x) = kx^2/2$, can be found from Ref. 6 with parameter $\alpha = 3m/2c^2$: $\omega = \omega_0 - (27X^2 \omega_0^3 / 16c^2) = \omega_0 [1 - (27V^2 / 16c^2)]$.

⁶Consider an oscillator consisting of a mass on a spring. The potential energy of the spring, $U = \frac{1}{2}kx^2$, is classical. We consider a kinetic energy term with a small quartic correction: $T = \frac{1}{2}m\dot{x}^2 + \frac{1}{4}\alpha\dot{x}^4$. If the amplitude of the oscillation is small, the solution can be found from a perturbation expansion. In terms of the amplitude of the oscillation, X , or in terms of the velocity amplitude V the resonant frequency, $\omega = \omega_0 - (9\alpha X^2 k^{3/2} / 8m^{5/2})$

$= \omega_0 - (9\alpha V^2 k^{1/2}/8m^{3/2}) = \omega_0 [1 - (9\alpha V^2/8m)]$. We can compare this result with the result of an intuitive calculation, in which the frequency of a classical harmonic oscillator is modified by a corrected mass term $\delta m = \alpha V^2/2$: $\omega = \sqrt{\frac{k}{m + \delta m}}$

$$= \sqrt{\frac{k}{m + \frac{1}{2}\alpha V^2}} = \omega_0 - \frac{\alpha V^2 k^{1/2}}{4m^{3/2}} = \omega_0 [1 - (\alpha V^2/4m)].$$

⁷I.A. Todoschenko, H. Alles, A. Babkin, A.Y. Parshin, and V. Tsepelin, *J. Low Temp. Phys.* **126**, 1449 (2002).

⁸M. Niemetz, W. Schoepe, J.T. Simola, and J.T. Tuoriniemi, *Physica B* **280**, 559 (2000).

⁹A.M. Guénault, V. Keith, C.J. Kennedy, and G.R. Pickett, *Phys. Rev. Lett.* **51**, 589 (1983).

¹⁰R. König, P. Esquinazi, and F. Pobell, *J. Low Temp. Phys.* **90**, 55 (1993).

¹¹G.B. Hess, *Phys. Rev. Lett.* **40**, 1191 (1978).

¹²H.J. Mikeska, *Phys. Rev.* **179**, 166 (1969).

¹³L. Kramer, *Phys. Rev.* **179**, 149 (1969).

¹⁴R.V. Duncan, G. Ahlers, and V. Steinberg, *Phys. Rev. Lett.* **60**, 1522 (1988).

¹⁵In superfluid ^4He there is also a velocity suppression of ρ_s , which is typically very small. Nevertheless, the nonlinearities arising in a superfluid ^4He oscillator have been seen by Hess (Ref. 11). The effects of superfluid density suppression due to flow in ^4He lead to a variety of effects near T_λ (Refs. 12–14).