Calculating Marty's Way: From Supergravity to LIGO

March 28, 2019

Marty Halpern Memorial

Zvi Bern

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ZB, J.J Carrasco, W.-M. Chen, A. Edison, H. Johansson, J. Parra-Martinez, R.Roiban, Mao Zeng, arXiv:1804.09311

ZB, C. Cheung, R. Roiban, C.H. Shen, M. Solon, M. Zeng, arXiv:1901.04424 and in preparation

Marty!

"Shut up and calculate"

— Marty Halpern

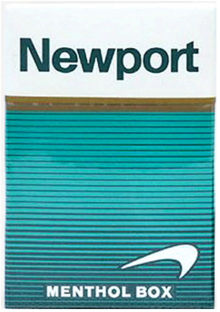
— Paul Dirac
— Richard Feynman
— David Mermin

If there ever was someone to attribute this to, it is Marty.

This is a gift: A basic way to think about theoretical physics.

Whatever success I've had stems from this gift from Marty.

"Blue collar theoretical physics": roll up your sleeves and calculate



Only tools Marty needed

“Calculate!”

Word of wisdom has guided me for the past 30 years.

Some ideas:

- **Duality between color and kinematics and double copy.**
- **Double copy as a means for understanding gravity.**

What is behind these ideas? How does it relate to standard approaches to gravity?

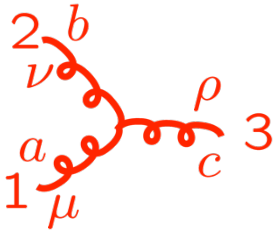
“Shut up and calculate”

1. **Applications of double copy to problem of UV divergence in quantum gravity.**
2. **Applications to LIGO physics.**

Some Basic Ideas About Gravity

Gravity Three Vertices

Standard Feynman diagram approach.



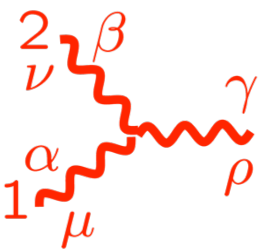
Three-gluon vertex:

$$V^{abc}_{3\,\mu\nu\sigma} = -gf^{abc}(\eta_{\mu\nu}(k_1-k_2)_\rho+\eta_{\nu\rho}(k_1-k_2)_\mu+\eta_{\rho\mu}(k_1-k_2)_\nu)$$

Three-graviton vertex:

$$k_i^2 = E_i^2 - \vec{k}_i^2 \neq 0$$

$$G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) =$$
$$\mathrm{sym}[-\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma})$$
$$+ P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma})$$
$$+ P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma})$$
$$+ 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})]$$



About 100 terms in three vertex

Naïve conclusion: Gravity is a nasty mess.

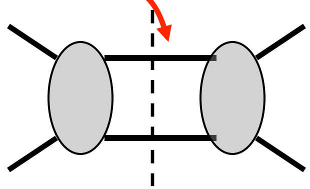
Generalized Unitarity Method

No Feynman rules; no need for virtual particles.

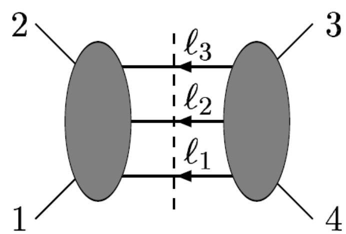
$$E^2 = \vec{p}^{\,2} + m^2$$ ← **on-shell**

ZB, Dixon, Dunbar and Kosower (1994)

Two-particle cut:

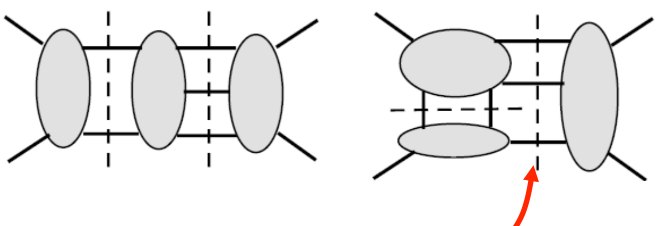


Three-particle cut:



- **Systematic assembly of complete amplitudes from other amplitudes.**
- **Works for any number of particles or loops.**

Generalized unitarity as a practical tool.



on-shell

ZB, Dixon and Kosower;
ZB, Morgan;
Britto, Cachazo, Feng;
Ossala,Pittau,Papadopoulos;
Ellis, Kunszt, Melnikov;
Forde; Badger
and many others

Reproduces Feynman diagrams, except intermediate steps of calculation based on gauge-invariant quantities.

Gravity Amplitudes

KLT (1985)

Kawai-Lewellen-Tye string relations in low energy limit:

gravity ↙ ↙ **gauge theory color ordered**

$$M_4^{\text{tree}}(1,2,3,4) = -i s_{12} A_4^{\text{tree}}(1,2,3,4)\, A_4^{\text{tree}}(1,2,4,3)\,,$$

$$M_5^{\text{tree}}(1,2,3,4,5) = i s_{12} s_{34} A_5^{\text{tree}}(1,2,3,4,5)\, A_5^{\text{tree}}(2,1,4,3,5) + i s_{13} s_{24} A_5^{\text{tree}}(1,3,2,4,5)\, A_5^{\text{tree}}(3,1,4,2,5)$$

Pattern gives explicit all-leg form.

Bern, Dixon, Perelstein, J.S. Rozowsky (1998)

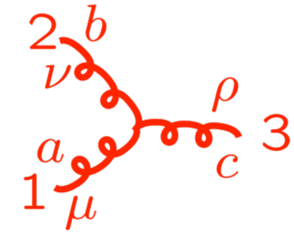
1. **Gravity is derivable from gauge theory. Standard Lagrangian methods offers no hint why this is possible.**
2. **It is very generally applicable.**

Duality Between Color and Kinematics

ZB, Carrasco, Johansson (2007)

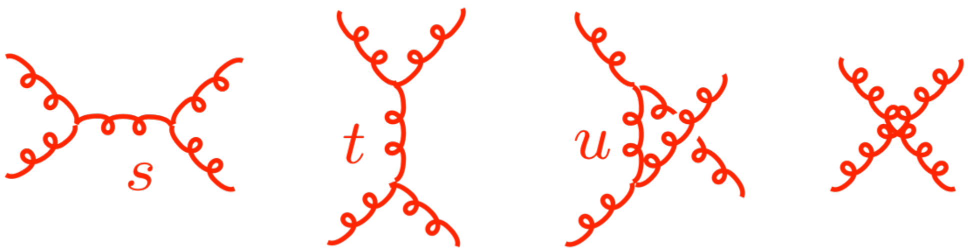
coupling constant → color factor → momentum dependent kinematic factor

$$-g f^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$



Color factors based on a Lie algebra: $[T^a, T^b] = i f^{abc} T^c$

Jacobi Identity $f^{a_1 a_2 b} f^{b a_4 a_3} + f^{a_4 a_2 b} f^{b a_3 a_1} + f^{a_4 a_1 b} f^{b a_2 a_3} = 0$



Use 1 = *s*/*s* = *t*/*t* = *u*/*u* to assign 4-point diagram to others.

$s = (k_1 + k_2)^2 \quad t = (k_1 + k_4)^2$

$u = (k_1 + k_3)^2$

$$\mathcal{A}_4^{\text{tree}} = g^2 \left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u} \right)$$

Color factors satisfy Jacobi identity:

Numerator factors satisfy similar identity:

$$c_u = c_s - c_t$$
$$n_u = n_s - n_t$$

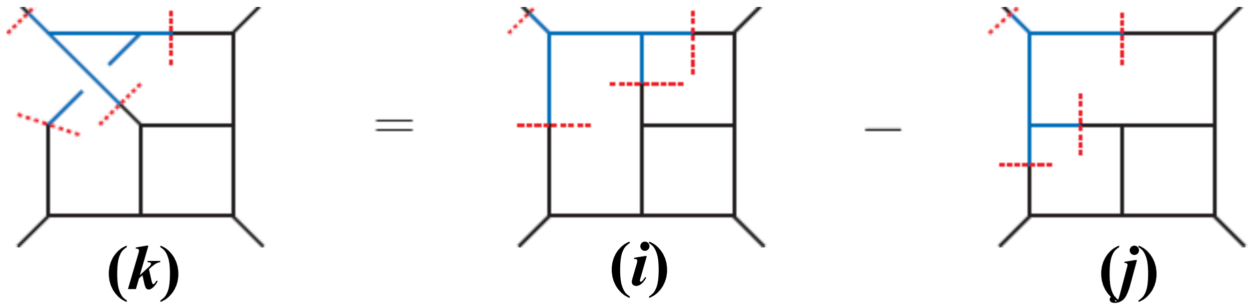
Proven at tree level

ZB, Carrasco, Johansson; Kiermaier; Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove; Cachazo, etc

Gravity Loop Integrands from Gauge Theory

BCJ

Ideas conjectured to generalize to loops:



color factor

$$c_k = c_i - c_j$$

$$n_k = n_i - n_j$$

kinematic numerator

If you have a set of duality satisfying numerators.

To get:

gauge theory → gravity theory

simply take

color factor → kinematic numerator

$$c_k \longrightarrow n_k$$

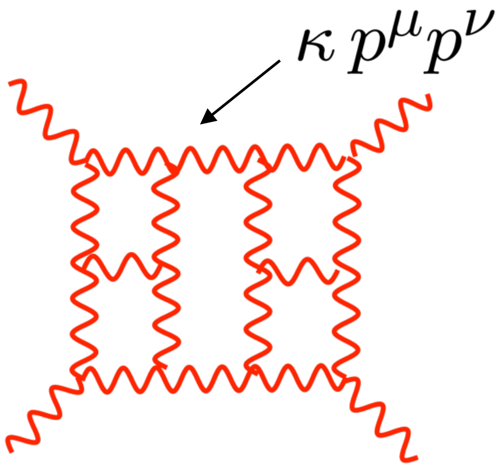
Gravity loop integrands follow from gauge theory!

UV Behavior of Gravity

“Shut up and Calculate”

UV Behavior of Gravity?

$\kappa = \sqrt{32\pi G_N}$ ⟵ Dimensionful coupling



Gravity: $$\int \prod_{i=1}^{L} \frac{d^D p_i}{(2\pi)^D} \frac{\cdots \kappa p_j^\mu p_j^\nu \cdots}{\text{propagators}}$$

Gauge theory: $$\int \prod_{i=1}^{L} \frac{d^D p_i}{(2\pi)^D} \frac{\cdots g p_j^\nu \cdots}{\text{propagators}}$$

- **Extra powers of loop momenta in numerator means integrals are badly behaved in the UV and must diverge at some loop order.**
- **Much more sophisticated power counting in supersymmetric theories but this is basic idea.**

- **With more supersymmetry expect better UV properties.**
- **Need to worry about "hidden cancellations".**
- ***N*** **= 8 supergravity *best* theory to study.**

$N = 8$ supergravity: Where is First $D = 4$ UV Divergence?

3 loops $N = 8$	Green, Schwarz, Brink (1982); Howe and Stelle (1989); Marcus and Sagnotti (1985)	✗
5 loops $N = 8$	Bern, Dixon, Dunbar, Perelstein, Rozowsky (1998); Howe and Stelle (2003,2009)	✗
6 loops $N = 8$	Howe and Stelle (2003)	✗
7 loops $N = 8$	Grisaru and Siegel (1982); Bossard, Howe, Stelle (2009);Vanhove; Björnsson, Green (2010); Kiermaier, Elvang, Freedman(2010); Ramond, Kallosh (2010); Biesert et al (2010); Bossard, Howe, Stelle, Vanhove (2011)	?
3 loops $N = 4$	Bossard, Howe, Stelle, Vanhove (2011)	✗
4 loops $N = 5$	Bossard, Howe, Stelle, Vanhove (2011)	✗
4 loops $N = 4$	Vanhove and Tourkine (2012)	✓
9 loops $N = 8$	Berkovits, Green, Russo, Vanhove (2009)	✗

"shut up and calculate"

ZB, Kosower, Carrasco, Dixon, Johansson, Roiban; ZB, Davies, Dennen, A. Smirnov, V. Smirnov; series of calculations.

This is what we are most interested in. (7 loops $N = 8$)

Weird structure. Anomaly-like behavior of divergence. (4 loops $N = 4$)

Retracted, but perhaps to be unretracted. (9 loops $N = 8$)

- **Track record of predictions from standard symmetries not great.**
- **Conventional wisdom holds that it will diverge sooner or later.**

Supersymmetry and Ultraviolet Divergences

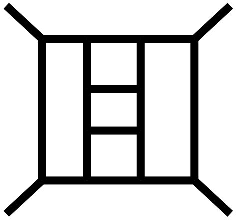
Bossard, Howe, Stelle; Elvang, Freedman, Kiermaier; Green, Russo, Vanhove ; Green and Björnsson ; Bossard , Hillmann and Nicolai; Ramond and Kallosh; Broedel and Dixon; Elvang and Kiermaier; Beisert, Elvang, Freedman, Kiermaier, Morales, Stieberger; Bossard, Howe, Stelle, Vanhove, etc

- **First quantized formulation of Berkovits' pure-spinor formalism.**
  Bjornsson and Green
- **Key point: *all* supersymmetry cancellations are exposed.**

Poor UV behavior, unless new types of cancellations between diagrams exist that are "not consequences of supersymmetry in any conventional sense".

Bjornsson and Green

- **$\mathcal{N} = 8$ sugra should diverge at 5 loops in $D = 24/5$.**
- **$\mathcal{N} = 8$ sugra should diverge at 7 loops in $D = 4$.**



Consensus agreement from all power-counting methods. including power counting within out unitarity approach.

"Shut up and calculate"

Scorecard on Symmetry Predictions

- $\mathcal{N} = 4$ sugra should diverge at 3 loops in $D = 4$. ✗
- $\mathcal{N} = 5$ sugra should diverge at 4 loops in $D = 4$. ✗
- Half maximal sugra diverges at 2 loops in $D = 5$. ✗
- $\mathcal{N} = 8$ sugra should diverge at 5 loops in $D = 24/5$. ✓
- $\mathcal{N} = 8$ sugra should diverge at 7 loops in $D = 4$. ? **key question**

ZB, Davies, Dennen (2012, 2014); ZB, Davies, Dennen, Huang(2012)
ZB, Carrasco, Chen, Edison, Johansson, Roiban, Parra-Martinez, Zeng (2018)

- UV cancellation of $\mathcal{N} = 5$ supergravity at 4 loops in $D = 4$ definite mystery. Clear problem with standard symmetry arguments.

Freedman, Kallosh and Yamada (2018)

What is the difference between $\mathcal{N} = 5$ and $\mathcal{N} = 8$? $D = 4$?

Goal is to provide definitive answers. Need to go to 7 loops!

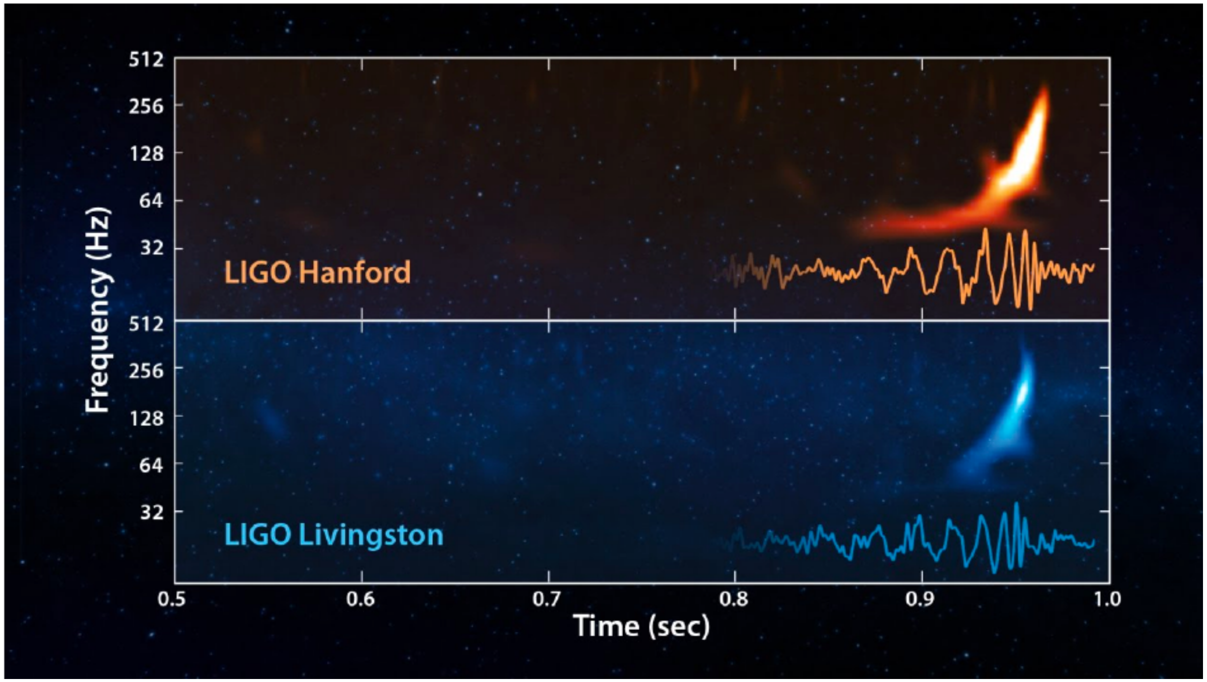
"Impossible" gravity calculations are pretty standard by now and shed nontrivial new light on UV properties.

Applications to LIGO Physics

"Shut up and Calculate"

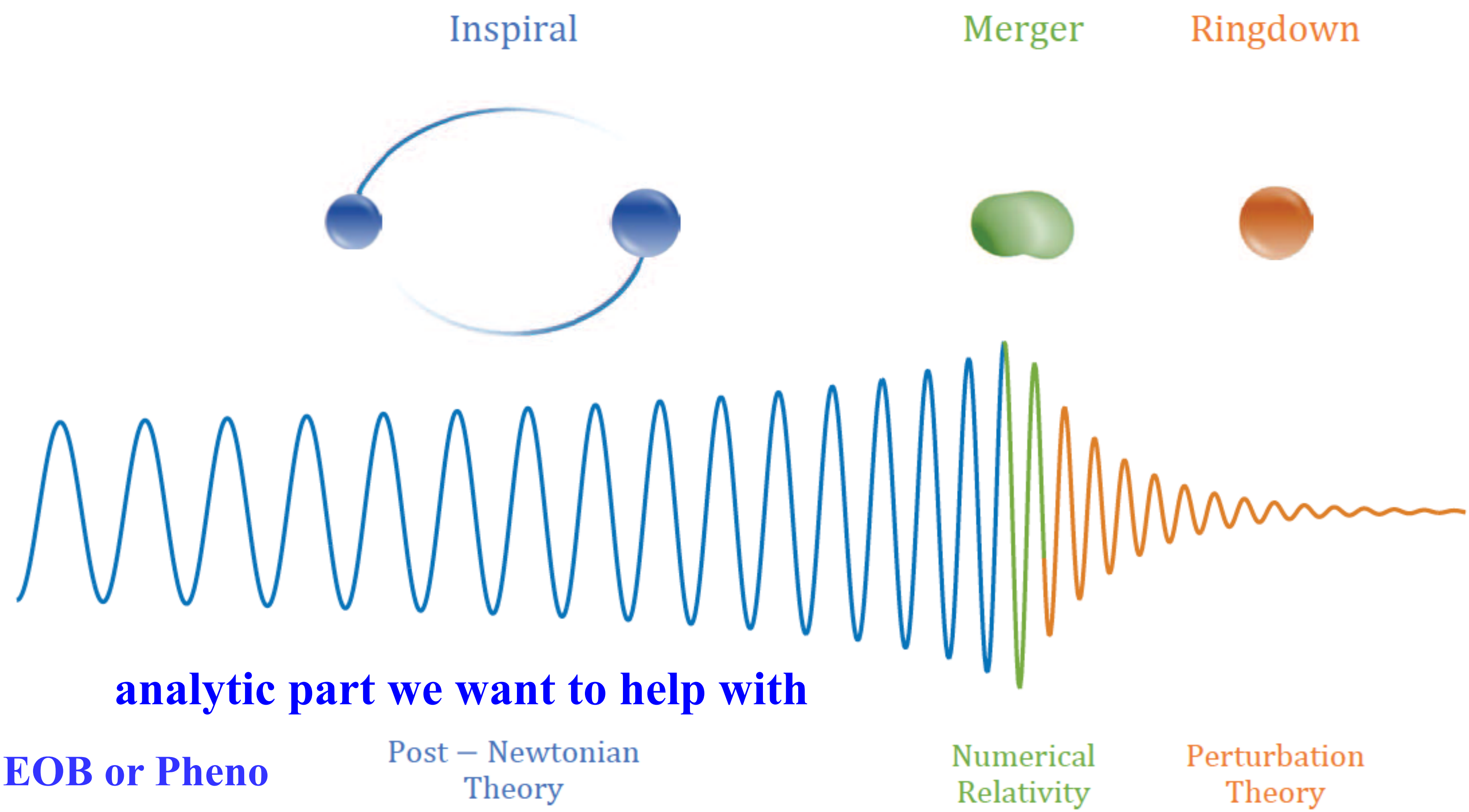
LIGO/Virgo and Templates

Era of gravitational wave astronomy begins!



Signal extracted and compared against ~250K templates computed from theory.

Goal: Improve on post-Newtonian Theory



analytic part we want to help with

PN + EOB or Pheno

Small errors accumulate. Need for high precision.

From Antelis and Moreno, arXiv:1610.03567

Post Newtonian Approximation

For orbital mechanics:

Expand in *G* and v^2

$$v^2 \sim \frac{GM}{r} \ll 1$$

virial theorem

In center of mass frame:

$$m = m_A + m_B, \quad \nu = \mu/M,$$

$$\mu = m_A m_B/m, \quad P_R = P \cdot \hat{R}$$

$$\frac{H}{\mu} = \frac{P^2}{2} - \frac{Gm}{R}$$

Newton

$$+ \frac{1}{c^2}\left\{ -\frac{P^4}{8} + \frac{3\,\nu\,P^4}{8} + \frac{Gm}{R}\left(-\frac{{P_R}^2\,\nu}{2} - \frac{3\,P^2}{2} - \frac{\nu\,P^2}{2}\right) + \frac{G^2m^2}{2R^2}\right\}$$

$$+ \ldots$$

1PN: Einstein, Infeld, Hoffmann

Hamiltonian known to 4PN order.

2PN: Ohta, Okamura, Kimura and Hiida.
3PN: Damour, Jaranowski and Schaefer; L. Blanchet and G. Faye.
4PN: Damour, Jaranowski and Schaefer (2014)

PN versus PM expansion for conservative two-body dynamics

From Buonanno Amplitudes 2018

$$\mathcal{L} = -Mc^2 + \underbrace{\frac{\mu v^2}{2} + \frac{GM\mu}{r}} + \frac{1}{c^2}\left[\dots\right] + \frac{1}{c^4}\left[\dots\right] + \dots$$

$$E(v) = -\frac{\mu}{2} v^2 + \cdots$$

non-spinning compact objects

		0PN	1PN	2PN	3PN	4PN	5PN	...
0PM:	1	v^2	v^4	v^6	v^8	v^{10}	v^{12}	...
1PM:		$1/r$	v^2/r	v^4/r	v^6/r	v^8/r	v^{10}/r	...
2PM:			$1/r^2$	v^2/r^2	v^4/r^2	v^6/r^2	v^8/r^2	...
3PM:				$1/r^3$	v^2/r^3	v^4/r^3	v^6/r^3	...
4PM:					$1/r^4$	v^2/r^4	v^4/r^4	...
...						...	...	

(credit: Justin Vines)

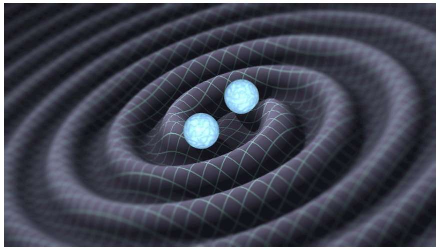
$$1 \to Mc^2, \qquad v^2 \to \frac{v^2}{c^2}, \qquad \frac{1}{r} \to \frac{GM}{rc^2}.$$

current known PN results — current known PM results — overlap between PN & PM results — unknown

- **PM results** (*Westfahl 79, Westfahl & Goller 80, Portilla 79-80, Bel et al. 81, Ledvinka et al. 10, Damour 16-17, Guevara 17, Vines 17, Bini & Damour 17-18, Vines in prep*)

Which problem to solve?

Some problems for (analytic) theorists:



1. **Spin.**
2. **Finite size effects.**
3. **Radiation.**
4. **High orders in perturbation theory.** ←

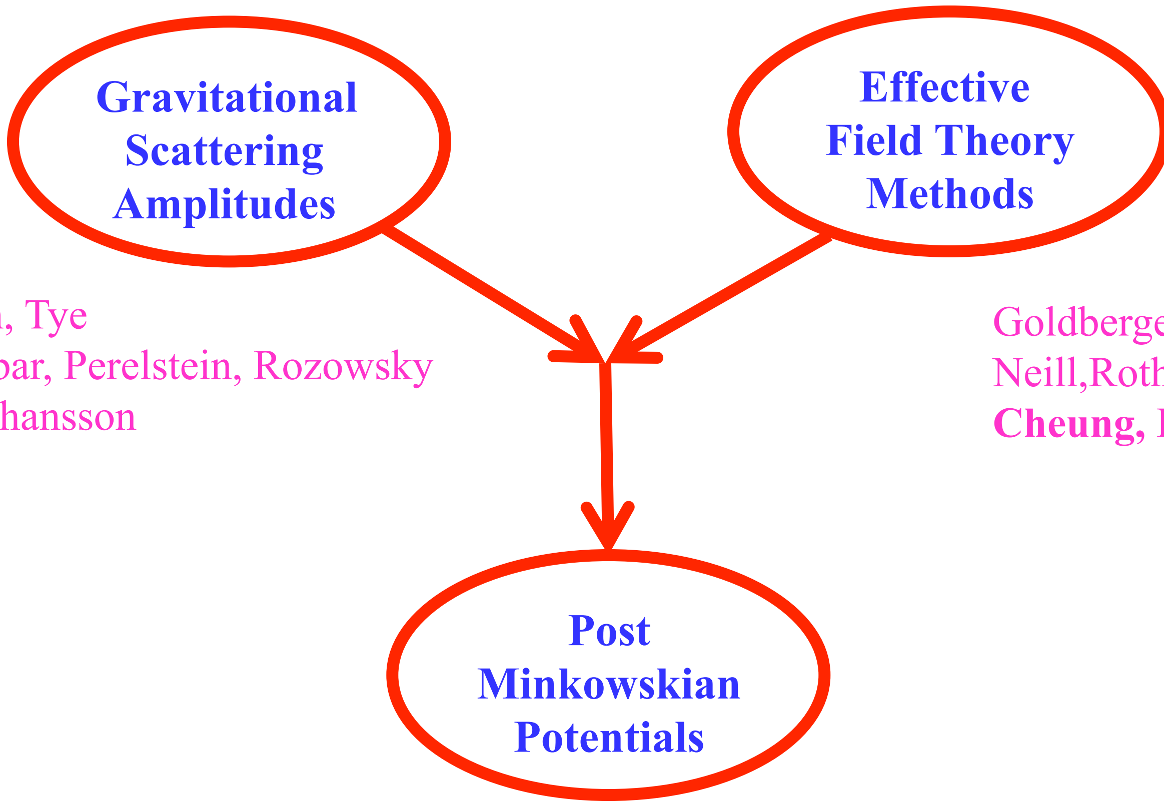
Which one to solve?

- **Needs to be extremely difficult using standard methods.**
- **Needs to be of direct interest to LIGO.**
- **Needs to be in a form that LIGO can use.**

3rd post-Minkowskian order 2-body Hamiltonian

Our Approach

ZB, Cheung, Roiban, Shen, Solon, Zeng



Kawai, Lewellen, Tye
ZB, Dixon, Dunbar, Perelstein, Rozowsky
ZB, Carrasco, Johansson

Goldberger, Rothstein
Neill,Rothstein
Cheung, Rothstein, Solon

Inefficient: Start with quantum theory and take $\hbar \to 0$

Efficient: Almost magical simplifications for gravity amplitudes.
EFT methods efficiently target pieces we want.

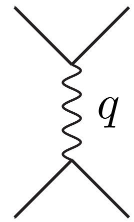
Will show efficiency wins.

Potentials and Amplitudes

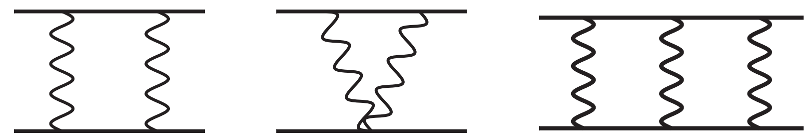
Iwasaki; Bjerrum-Bohr, Donoghue, Vanhove; Neill, Rothstein
Bjerrum-Bohr, Damgaard, Festuccia, Planté. Vanhove

Tree-level: Fourier transform gives classical potential.

$$V(r) = \int \frac{d^3q}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} A^{\text{tree}}(\mathbf{q})$$



At higher orders things quickly become less obvious:



- **What you learned in grad school on $\hbar$ counting is wrong. Loops can have classical pieces.**
- **Double counting and iteration.**
- **$1/\hbar$ scaling of loop amplitudes.**
- **Non-uniqueness of potential.**
- **Cross terms between $1/\hbar$ and $\hbar$**

$$e^{iS_{\text{classical}}/\hbar}$$

$1/\hbar^L$ at L loops

Piece of loops are classical: Our task is to extract these pieces.

We harness EFT to clean up confusion

EFT Matching

Goldberger and Rothstein
Neill, Rothstein
Cheung, Rothstein, Solon (2018)

full Einstein's theory
(complicated)

Amplitude methods
double copy

tree amplitude

$\hbar \to 0$

generalized
unitarity

loop integrand

loop
integration

GR loop amplitude

effective theory
(simpler)

build
ansatz

potential

Feynman
diagrams

loop integrand

loop
integration

EFT loop amplitude

identical
physics

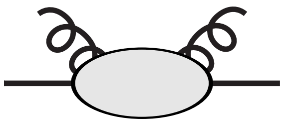
GR loop amplitude $=$ **EFT loop amplitude**

Roundabout, but efficiently determines potential

Gauge-Theory Building Blocks for 2 PM Gravity

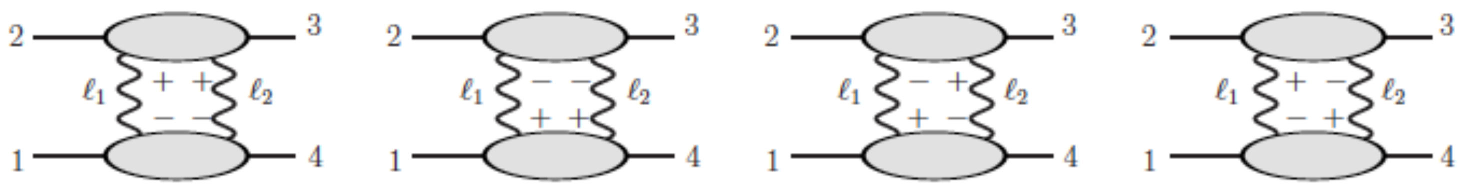
$$A^{\text{tree}}(1^s, 2^+, 3^+, 4^s) = i\frac{m_1^2[23]}{\langle 23\rangle t_{12}}$$

$$A^{\text{tree}}(1^s, 2^+, 3^-, 4^s) = -i\frac{\langle 3|1|2]^2}{\langle 23\rangle[23]t_{12}}$$



color-ordered gauge-theory tree amplitudes

- **This is all you need for 2 PM.**
- **Scaling with number of loops is very good.**



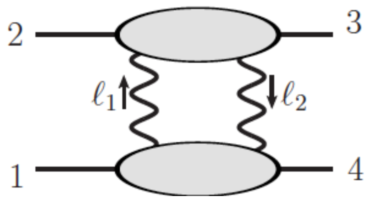
$$C_{\text{YM}} = 2\left(\frac{\mathcal{E}^2 + \mathcal{O}^2}{s_{23}^2} + m_1^2 m_2^2\right)\frac{1}{t_{1\ell_1}\, t_{2\ell_1}}$$

gauge theory integrand

$$\mathcal{E}^2 = \frac{1}{4}\Big[-t_{12}s_{23} + s_{23}t_{1\ell_1} - s_{23}t_{2\ell_1} + 2t_{1\ell_1}t_{2\ell_1}\Big]^2$$

$$\mathcal{O}^2 = \mathcal{E}^2 - (s_{23}m_1^2 + s_{23}t_{1\ell_1} + t_{1\ell_1}^2)(s_{23}m_2^2 - s_{23}t_{2\ell_1} + t_{2\ell_1}^2)$$

One loop gravity warmup



Apply unitarity and KLT relations.
Import gauge-theory results.

$$C_{\rm GR} = 2\left[\frac{1}{t^4}\left(\mathcal{E}^4 + \mathcal{O}^4 + 6\mathcal{E}^2\mathcal{O}^2\right) + m_1^4 m_2^4\right]\left[\frac{1}{t_{1\ell_1}} + \frac{1}{t_{4\ell_1}}\right]\left[\frac{1}{t_{2\ell_1}} + \frac{1}{t_{3\ell_1}}\right]$$

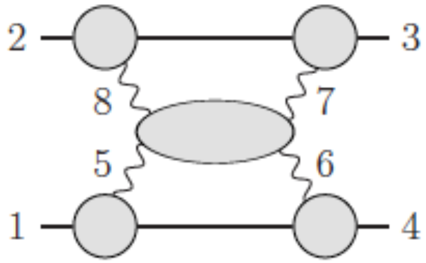
- **Same building blocks as gauge theory!**
- **Double copy is visible even though we have removed dilaton and axion.**

We can extract classical scattering angles or potentials following literature Damour; Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove; Cheung, Rothstein, Solon

This is 2nd PM order

$s_{23} = (p_2 + p_3)^2$
$t_{ij} = 2p_i \cdot p_j$

Two Loops for 3 PM



ZB, Cheung, Shen, Roiban, Solon, Zeng

- **Use KLT and sum over helicities**
- **Very similar to one loop**

$$C^{\text{H-cut}} = 2i\left[\frac{1}{(p_5 - p_8)^2} + \frac{1}{(p_5 + p_7)^2}\right] \times \left[s_{23}^2 m_1^4 m_2^4 + \frac{1}{s_{23}^6}\sum_{i=1,2}\left(\mathcal{E}_i^4 + \mathcal{O}_i^4 + 6\mathcal{O}_i^2\mathcal{E}_i^2\right)\right]$$

$$\mathcal{E}_1^2 = \frac{1}{4}s_{23}^2(t_{18}t_{25} - t_{12}t_{58})^2, \qquad \mathcal{O}_1^2 = \mathcal{E}_1^2 - m_1^2 m_2^2 s_{23}^2 t_{58}^2,$$

$$\mathcal{E}_2^2 = \frac{1}{4}s_{23}^2(t_{17}t_{25} - t_{12}t_{57} - s_{23}(t_{17} + t_{57}))^2,$$

$$\mathcal{O}_2^2 = \mathcal{E}_2^2 - m_1^2 m_2^2 s_{23}^2 t_{57}^2.$$

- **Double copy is visible.**
- **Remarkably simple, given it is two-loop gravity.**
- **Other cuts somewhat more complicated, but straightforward**

Integration + Extraction of Potential

To integrate follow methods of Cheung, Rothstein and Solon.

- **Efficiently targets the classical pieces we want.**
- **Integrals reduce via residues to 3 dimensional integrals.**
- **Incorporates matching to effective field theory.**
- **Good scaling with perturbative order.**

Checks on integrals using standard tools of QCD:

- **Mellin-Barnes integration.** V. Smirnov; Czakon
- **Sector decompostion.** Binoth and Heinrich, A. Smirnov
- **Integration by parts.** K. G. Chetyrkin and F. V. Tkachov, Laporta; A. Smirnov; Maierhöfer, Usovitsch, Uwer
- **Differential equations.** ZB, Dixon, Kosower; Remiddi and Gehrmann
- **Method of regions.** Beneke, V. Smirnov; A. Smirnov.

Amplitude in Classical Potential Limit

ZB, Cheung, Roiban, Shen, Solon, Zeng (2019)

Classical limit. The O(G^3) or 3PM terms are:

rapidity

$$\mathcal{M}_3 = \frac{\pi G^3 \nu^2 m^4 \log \boldsymbol{q}^2}{6\gamma^2 \xi}\left[3 - 6\nu + 206\nu\sigma - 54\sigma^2 + 108\nu\sigma^2 + 4\nu\sigma^3 - \frac{48\nu\left(3 + 12\sigma^2 - 4\sigma^4\right)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}} - \frac{18\nu\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)}{(1+\gamma)(1+\sigma)}\right] + \frac{8\pi^3 G^3 \nu^4 m^6}{\gamma^4 \xi}\left[3\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)F_1 - 32 m^2 \nu^2 \left(1-2\sigma^2\right)^3 F_2\right]$$

$$m = m_A + m_B, \qquad \mu = m_A m_B/m, \qquad \nu = \mu/m, \qquad \gamma = E/m,$$
$$\xi = E_1 E_2/E^2, \qquad E = E_1 + E_2, \qquad \sigma = p_1 \cdot p_2/m_1 m_2,$$

F_1 and F_2 IR divergent iteration terms that don't affect potential.

Two loop gravity.

Simplicity of result is remarkable!

Conservative 3PM Potential

ZB, Cheung, Roiban, Shen, Solon, Zeng

Follow EFT strategy:

The 3PM Hamiltonian:

$$H(\boldsymbol{p},\boldsymbol{r}) = \sqrt{\boldsymbol{p}^2 + m_1^2} + \sqrt{\boldsymbol{p}^2 + m_2^2} + V(\boldsymbol{p},\boldsymbol{r})$$

$$V(\boldsymbol{p},\boldsymbol{r}) = \sum_{i=1}^{\infty} c_i(\boldsymbol{p}^2)\left(\frac{G}{|\boldsymbol{r}|}\right)^i,$$

$$c_1 = \frac{\nu^2 m^2}{\gamma^2\xi}\left(1-2\sigma^2\right), \qquad c_2 = \frac{\nu^2 m^3}{\gamma^2\xi}\left[\frac{3}{4}\left(1-5\sigma^2\right) - \frac{4\nu\sigma\left(1-2\sigma^2\right)}{\gamma\xi} - \frac{\nu^2(1-\xi)\left(1-2\sigma^2\right)^2}{2\gamma^3\xi^2}\right],$$

$$\begin{aligned} c_3 = \frac{\nu^2 m^4}{\gamma^2\xi}\Bigg[&\frac{1}{12}\left(3-6\nu+206\nu\sigma-54\sigma^2+108\nu\sigma^2+4\nu\sigma^3\right) - \frac{4\nu\left(3+12\sigma^2-4\sigma^4\right)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}} \\ &- \frac{3\nu\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)}{2(1+\gamma)(1+\sigma)} - \frac{3\nu\sigma\left(7-20\sigma^2\right)}{2\gamma\xi} - \frac{\nu^2\left(3+8\gamma-3\xi-15\sigma^2-80\gamma\sigma^2+15\xi\sigma^2\right)\left(1-2\sigma^2\right)}{4\gamma^3\xi^2} \\ &+ \frac{2\nu^3(3-4\xi)\sigma\left(1-2\sigma^2\right)^2}{\gamma^4\xi^3} + \frac{\nu^4(1-2\xi)\left(1-2\sigma^2\right)^3}{2\gamma^6\xi^4}\Bigg], \end{aligned}$$

$$m = m_A + m_B, \qquad \mu = m_A m_B/m, \qquad \nu = \mu/m, \qquad \gamma = E/m,$$

$$\xi = E_1 E_2/E^2, \qquad E = E_1 + E_2, \qquad \sigma = p_1 \cdot p_2/m_1 m_2,$$

Checks

ZB, Cheung, Roiban, Shen, Solon, Zeng

Primary check:

Compare to 4PN Hamiltonian of Damour, Jaranowski, Schäfer

Need canonical transformation:

$$(\boldsymbol{r},\boldsymbol{p}) \to (\boldsymbol{R},\boldsymbol{P}) = (A\,\boldsymbol{r} + B\,\boldsymbol{p}, C\,\boldsymbol{p} + D\,\boldsymbol{r})$$

$$A = 1 - \frac{Gm\nu}{2|\boldsymbol{r}|} + \cdots\,, \quad B = \frac{G(1-2/\nu)}{4m|\boldsymbol{r}|}\boldsymbol{p}\cdot\boldsymbol{r} + \cdots$$

$$C = 1 + \frac{Gm\nu}{2|\boldsymbol{r}|} + \cdots\,, \quad D = -\frac{Gm\nu}{2|\boldsymbol{r}|^3}\boldsymbol{p}\cdot\boldsymbol{r} + \cdots\,,$$

For overlap terms of our Hamiltonian equivalent to 4PN Hamitonian. Explicit canonical transformation found.

4 PN Hamiltonian

Damour, Jaranowski, Schaefer

$$\mathbf{n} = \hat{\mathbf{r}}$$

$$\widehat{H}_{\rm N}\,(\mathbf{r},\mathbf{p}) = \frac{\mathbf{p}^2}{2} - \frac{1}{r},$$

$$c^2\,\widehat{H}_{\rm 1PN}\,(\mathbf{r},\mathbf{p}) = \frac{1}{8}(3\nu-1)(\mathbf{p}^2)^2 - \frac{1}{2}\Big\{(3+\nu)\mathbf{p}^2 + \nu(\mathbf{n}\cdot\mathbf{p})^2\Big\}\frac{1}{r} + \frac{1}{2r^2},$$

$$c^4\,\widehat{H}_{\rm 2PN}\,(\mathbf{r},\mathbf{p}) = \frac{1}{16}\left(1-5\nu+5\nu^2\right)(\mathbf{p}^2)^3 + \frac{1}{8}\Big\{\left(5-20\nu-3\nu^2\right)(\mathbf{p}^2)^2 - 2\nu^2(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 - 3\nu^2(\mathbf{n}\cdot\mathbf{p})^4\Big\}\frac{1}{r}$$
$$+ \frac{1}{2}\Big\{(5+8\nu)\mathbf{p}^2 + 3\nu(\mathbf{n}\cdot\mathbf{p})^2\Big\}\frac{1}{r^2} - \frac{1}{4}(1+3\nu)\frac{1}{r^3},$$

$$c^6\,\widehat{H}_{\rm 3PN}\,(\mathbf{r},\mathbf{p}) = \frac{1}{128}\left(-5+35\nu-70\nu^2+35\nu^3\right)(\mathbf{p}^2)^4 + \frac{1}{16}\Big\{\left(-7+42\nu-53\nu^2-5\nu^3\right)(\mathbf{p}^2)^3$$
$$+ (2-3\nu)\nu^2(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 + 3(1-\nu)\nu^2(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 - 5\nu^3(\mathbf{n}\cdot\mathbf{p})^6\Big\}\frac{1}{r}$$
$$+ \Big\{\frac{1}{16}\left(-27+136\nu+109\nu^2\right)(\mathbf{p}^2)^2 + \frac{1}{16}(17+30\nu)\nu(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 + \frac{1}{12}(5+43\nu)\nu(\mathbf{n}\cdot\mathbf{p})^4\Big\}\frac{1}{r^2}$$
$$+ \Bigg\{\bigg(-\frac{25}{8} + \Big(\frac{\pi^2}{64} - \frac{335}{48}\Big)\nu - \frac{23\nu^2}{8}\bigg)\mathbf{p}^2 + \Big(-\frac{85}{16} - \frac{3\pi^2}{64} - \frac{7\nu}{4}\Big)\nu(\mathbf{n}\cdot\mathbf{p})^2\Bigg\}\frac{1}{r^3} + \Bigg\{\frac{1}{8} + \Big(\frac{109}{12} - \frac{21}{32}\pi^2\Big)\nu\Bigg\}\frac{1}{r^4},$$

G^4

4 PN Hamiltonian

Damour, Jaranowski, Schaefer

$$\mathbf{n} = \hat{\mathbf{r}}$$

$$
\begin{aligned}
c^8\,\widehat{H}^{\mathrm{local}}_{4\mathrm{PN}}(\mathbf{r},\mathbf{p}) = {} & \left(\frac{7}{256} - \frac{63}{256}\nu + \frac{189}{256}\nu^2 - \frac{105}{128}\nu^3 + \frac{63}{256}\nu^4\right)(\mathbf{p}^2)^5 \\
& + \Bigg\{ \frac{45}{128}(\mathbf{p}^2)^4 - \frac{45}{16}(\mathbf{p}^2)^4\,\nu + \left(\frac{423}{64}(\mathbf{p}^2)^4 - \frac{3}{32}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^3 - \frac{9}{64}(\mathbf{n}\cdot\mathbf{p})^4(\mathbf{p}^2)^2\right)\nu^2 \\
& + \left(-\frac{1013}{256}(\mathbf{p}^2)^4 + \frac{23}{64}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^3 + \frac{69}{128}(\mathbf{n}\cdot\mathbf{p})^4(\mathbf{p}^2)^2 - \frac{5}{64}(\mathbf{n}\cdot\mathbf{p})^6\mathbf{p}^2 + \frac{35}{256}(\mathbf{n}\cdot\mathbf{p})^8\right)\nu^3 \\
& + \left(-\frac{35}{128}(\mathbf{p}^2)^4 - \frac{5}{32}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^3 - \frac{9}{64}(\mathbf{n}\cdot\mathbf{p})^4(\mathbf{p}^2)^2 - \frac{5}{32}(\mathbf{n}\cdot\mathbf{p})^6\mathbf{p}^2 - \frac{35}{128}(\mathbf{n}\cdot\mathbf{p})^8\right)\nu^4 \Bigg\}\frac{1}{r} \\
& + \Bigg\{ \frac{13}{8}(\mathbf{p}^2)^3 + \left(-\frac{791}{64}(\mathbf{p}^2)^3 + \frac{49}{16}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 - \frac{889}{192}(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 + \frac{369}{160}(\mathbf{n}\cdot\mathbf{p})^6\right)\nu \\
& + \left(\frac{4857}{256}(\mathbf{p}^2)^3 - \frac{545}{64}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 + \frac{9475}{768}(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 - \frac{1151}{128}(\mathbf{n}\cdot\mathbf{p})^6\right)\nu^2 \\
& + \left(\frac{2335}{256}(\mathbf{p}^2)^3 + \frac{1135}{256}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 - \frac{1649}{768}(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 + \frac{10353}{1280}(\mathbf{n}\cdot\mathbf{p})^6\right)\nu^3 \Bigg\}\frac{1}{r^2} \\
& + \Bigg\{ \frac{105}{32}(\mathbf{p}^2)^2 + \Bigg(\left(\frac{2749\pi^2}{8192} - \frac{589189}{19200}\right)(\mathbf{p}^2)^2 + \left(\frac{63347}{1600} - \frac{1059\pi^2}{1024}\right)(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 + \left(\frac{375\pi^2}{8192} - \frac{23533}{1280}\right)(\mathbf{n}\cdot\mathbf{p})^4\Bigg)\nu \\
& + \Bigg(\left(\frac{18491\pi^2}{16384} - \frac{1189789}{28800}\right)(\mathbf{p}^2)^2 + \left(-\frac{127}{3} - \frac{4035\pi^2}{2048}\right)(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 + \left(\frac{57563}{1920} - \frac{38655\pi^2}{16384}\right)(\mathbf{n}\cdot\mathbf{p})^4\Bigg)\nu^2 \\
& + \left(-\frac{553}{128}(\mathbf{p}^2)^2 - \frac{225}{64}(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 - \frac{381}{128}(\mathbf{n}\cdot\mathbf{p})^4\right)\nu^3 \Bigg\}\frac{1}{r^3} \\
& + \Bigg\{ \frac{105}{32}\mathbf{p}^2 + \Bigg(\left(\frac{185761}{19200} - \frac{21837\pi^2}{8192}\right)\mathbf{p}^2 + \left(\frac{3401779}{57600} - \frac{28691\pi^2}{24576}\right)(\mathbf{n}\cdot\mathbf{p})^2\Bigg)\nu \\
& + \Bigg(\left(\frac{672811}{19200} - \frac{158177\pi^2}{49152}\right)\mathbf{p}^2 + \left(\frac{110099\pi^2}{49152} - \frac{21827}{3840}\right)(\mathbf{n}\cdot\mathbf{p})^2\Bigg)\nu^2 \Bigg\}\frac{1}{r^4} \quad \longleftarrow G^4 \\
& + \Bigg\{ -\frac{1}{16} + \left(\frac{6237\pi^2}{1024} - \frac{169199}{2400}\right)\nu + \left(\frac{7403\pi^2}{3072} - \frac{1256}{45}\right)\nu^2 \Bigg\}\frac{1}{r^5}. \quad \longleftarrow G^5
\end{aligned}
$$

After canonical transformation we match all but G^4 and G^5 terms

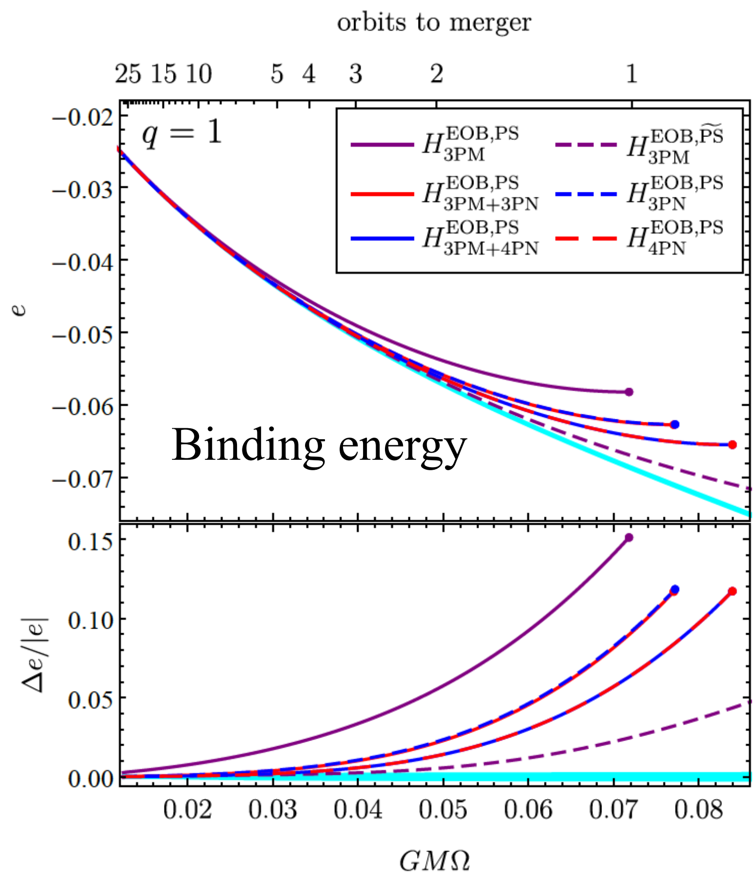
Mess is partly due to their gauge choice.

Ours is all orders in *p* at G^3

Tests of Our 3PM Hamiltonian for LIGO

Antonelli, Buonanno, Steinhoff, van de Meent, and Vines, arXiv:1901.07102

(8 days after our paper!)



Fed into EOB formalism.

Test against numerical relativity.

Note: Not conclusive, e. g. radiation not taken into account.

Winning curve is based on feeding 3PM through machinery.

numerical relativity taken as truth

"This rather encouraging result motivates a more comprehensive study…"

3PM + 4PN fed into EOB ➔ Most advanced 2 body Hamiltonian

Outlook for Gravitational Wave Physics

- **Methods are far from exhausted.**
- **Even more efficient methods seem likely.**
- **Methods should scale well to higher orders.**

Natural future questions to investigate:

- **Higher orders. Resummation in *G*.**
- **Radiation.**
- **Spin.**
- **Finite size effects.**

Summary

- Marty's basic lesson: "shut up and calculate".
- Remarkable connection between gauge and gravity theories:
  — color ⟷ kinematics.
  — gravity ~ (gauge theory)^2
- Double-copy idea gives us a powerful new way to think about gravity. We know this because we calculate.
- High loop orders in (super)gravity now common. 5 loops.
- Obtained the 3PM conservative 2-body potential for LIGO.
- Methods nowhere close to exhausted.

Expect many more calculations in the coming years!