An upper bound to time-averaged space-charge limited diode currents

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The Child–Langmuir law limits the steady-state current density across a one-dimensional planar diode. While it is known that the peak current density can surpass this limit when the boundary conditions vary in time, it remains an open question of whether the average current can violate the Child–Langmuir limit under time-dependent conditions. For the case where the applied voltage is constant but the electric field at the cathode is allowed to vary in time, one-dimensional particle-in-cell simulations suggest that such a violation is impossible. Although a formal proof is not given, an upper bound on the time-averaged current density is offered. © *2010 American Institute of Physics*. [doi:10.1063/1.3503661]

The Child–Langmuir law[1] gives the space-charged limited current in the classical problem of a one-dimensional (1D) diode

$$J_{\mathrm{CL}} = \frac{4}{9}\epsilon_0 \sqrt{\frac{2q}{m}}\frac{V^{3/2}}{d^2}.$$

Many interesting generalizations of this effect have been considered, particularly with respect to geometry,[2–4] nonzero injection velocities,[5] and relativistic[6,7] and quantum effects.[8,9] Time-dependent problems have also been studied, for example short current pulses[10,11] and time-varying voltage drops to control startup transients.[12,13] However, it has not been shown that the space-charged limited current cannot be exceeded on average under time-varying boundary conditions. We approach this problem first through numerical investigations, which suggest that the Child–Langmuir limit pertains even under these relaxed assumptions. Although we cannot prove rigorously the Child–Langmuir limit, we do prove an upper bound for the average current, and we pose as a conjecture that the rigorous bound is in fact the Child–Langmuir limit. Consider a 1D diode with a voltage gain V between the cathode and the anode. Electrons are injected at rest from the cathode and then accelerated across to the anode where they exit the diode. In the steady-state problem, the current is maximized when the electric field vanishes at the cathode due to the accumulated space charge. At that point, no further current can be extracted from the cathode. However, consider the case where the current is limited at the cathode through other effects, prior to the space charge limit. In this case, it is assumed that the electric field can fall below zero (push electrons away from the cathode) but cannot be larger than zero (push electrons back into the cathode). The question is whether this flexibility can allow more than the space charge limited current on average. Surprisingly, even though numerical simulations suggest that this flexibility does not allow more current, a rigorous proof remains elusive. In one dimension, the electric field in the diode is given by

$$E(x) = E_b + \frac{1}{2\epsilon_0}\left[\int_0^x \rho(x')dx' - \int_x^d \rho(x')dx'\right], \tag{1}$$

where $x=0$ at the cathode and $x=d$ at the anode. E_b is constant across the diode and is determined by the applied voltage drop. Integrating Eq. (1) by parts to solve for E_b results in

$$E(x) = \frac{V}{d} - \frac{Q}{\epsilon_0}\left(1 - \frac{x_q}{d}\right) + \frac{1}{\epsilon_0}\int_0^x \rho(x')dx', \tag{2}$$

where Q is the total charge in the diode and $x_q \equiv (1/Q)\int_0^d x'\rho(x')dx'$ is the "center of charge." A simple upper limit on the average current density is given by the maximum charge allowed in the diode at any one time divided by the fastest possible transit time of an electron

$$\bar{J}_{cl} \le \frac{Q_{\max}}{\tau_{\min}}. \tag{3}$$

In order to find a limit on the total charge in the diode, consider two physical constraints on the charge density $\rho(x)$. The first constraint is that $\rho(x)$ must satisfy the condition that $qE(x=0)\ge 0$

$$\int_0^d \rho(x')\left(1 - \frac{x'}{d}\right)dx' \le \frac{\epsilon_0 V}{d}. \tag{4}$$

The second constraint can be found by noting that an element of charge that is injected at the cathode will expand due to its own space charge as it moves toward the anode. If a thin, uniform element of charge Q and width ξ is injected from the cathode at $t=0$ then its charge density at a subsequent times is given by

$$\rho = \frac{Q}{[\xi + (qQ/2m\epsilon_0)t^2]}. \tag{5}$$

Equation (5) follows from the difference in acceleration between an electron at the very front of the charge element, and one at the very back, $a_f - a_b = (e/m)(Q/\epsilon_0)$. No fluid elements cross paths $\{[d(qE)/dx] = q\rho > 0;\ v(x=0)=0\}$ so Q will remain constant in time, and if the charge pulse initially has uniform charge density, then the charge density will remain uniform as it expands, by the continuity equation. The maximum possible charge density at position x occurs when $\xi=0$ and t is given by the fastest possible transit time, $t=t_{\min}(x)$. The time it takes for an electron to get to $x(t_{\min})$ is discussed in the next section. This results in a second constraint on the maximum charge density

$$\rho(x) \le \rho_{\max}(x) = \frac{2m\epsilon_0}{qt_{\min}^2(x)}. \tag{6}$$

It is clear from Eq. (4) that the charge in the diode is maximized when all the charge is placed as far toward $x=d$ as possible. Therefore, the maximum charge can be found by integrating $\rho_{\max}(x)$ from $x=d$ to the point $x*$, where Eq. (4) is violated

$$\int_{x*}^{d} \rho_{\max}(x')\left(1-\frac{x'}{d}\right)dx' = \frac{\epsilon_0 V}{d}, \tag{7}$$

$$Q_{\max} = \int_{x*}^{d} \rho_{\max}(x')dx'. \tag{8}$$

Using the transit time information from the following section, this gives an upper limit on the total charge in the diode of $1.44 \cdot Q_{\mathrm{CL}}$ where $Q_{\mathrm{CL}}=(4/3)(\epsilon_0 V/d)$ is the total charge in the diode in the steady-state case. An initial limit on $t_{\min}(x)$, the transit time to position x, comes from Gauss's law and the boundary conditions of the diode. Once this initial limit is established, it can be used to put limits on the charge density, $\rho_{\max}(x)$. Using the limits on the charge density in Eq. (2), we can further limit $t_{\min}(x)$ and repeat this in an iterative process that quickly converges. To start, note that the shortest transit time to position x is limited by the electric field

$$m\frac{dv}{dt} = qE_{\max}(x) \tag{9}$$

and $E_{\max}$ is constrained by Gauss' law.

$$\frac{d(qE)}{dx} = \frac{q\rho}{\epsilon_0} \ge 0. \tag{10}$$

It follows from Eq. (10) that the magnitude of E can only increase with x. Combined with the constraint that there is a constant voltage drop across the diode, this sets an upper limit on the electric field, $|E_{\max}|=(V/d-x)$. Integrating electron motion across the diode using $E_{\max}$ will give a lower limit for $t_{\min}$ that we use to find a preliminary form of $\rho_{\max}(x)$. Using this initial $\rho_{\max}(x)$ combined with Eq. (2) can give a stricter limit on $E_{\max}$ and $t_{\min}$, which in turn gives a stricter limit on $\rho_{\max}(x)$. The process can be repeated iteratively, and quickly reaches the asymptotic limit

$$t_{\min} \ge 0.62 \cdot T_{cl}, \tag{11}$$

where $T_{cl} \equiv Q_{\mathrm{CL}}/J_{\mathrm{CL}}=3d\sqrt{m/2qV}$. Combining the results from the previous sections gives an upper limit on the average current density

$$\bar{J}_{\max} \le 2.45 \cdot J_{cl}. \tag{12}$$

This may well be a large overestimate of the achievable average current. In 1D particle-in-cell (PIC) simulations using a simple code developed for this purpose, sample input currents were tested. Several input currents of dissimilar functional forms were able to closely approach the steady-state limit on current density but even after optimizing some free parameters associated with the input currents, none was able to surpass it. This does not prove anything because the parameter space of input currents that we tested was far from exhaustive. However, it leads us to conjecture that there is a hard limit on the average current density in a 1D planar diode as $t\to\infty$ that is equal to the Child–Langmuir limit. In conclusion, although we offer an upper bound to the maximum average current emitted by a planar diode, it remains to prove rigorously that the Child–Langmuir law is obeyed on average as well, as suggested by our numerical simulations.

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