

# On the density oscillations of a warm particle bunch

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The density oscillations of warm particle bunches is investigated theoretically. Two different mathematical approaches are employed to derive the basic equation describing density oscillations; one is a fluid approach and the second is a more general Green's function formulation. The motion is analyzed in first-order perturbation theory where it is shown, under the assumption of no degeneracy, that there are only stable oscillations. Second-order perturbation theory gives damping of the motion. The perturbation theory is examined, and a criterion is exhibited for its proper use. Thus, when the resistivity is small enough (but nonzero) the motion is stable, but when the resistivity is large the motion is essentially unstable with a growth rate which is that of an unbunched beam. The criterion is approximately evaluated using a model for a bunched beam.

## I. INTRODUCTION

Present heavy-ion-fusion schemes require the manipulation and acceleration of intense particle bunches. The radio frequency (rf) linear accelerator system uses storage rings to multiply current, while the induction linac accelerates significant currents directly to the target. Common to both methods is the need for the stability of intense bunches of particles. This has been the subject of much research.<sup>1-4</sup>

A simple estimate of the growth length of a longitudinal resistive instability in an induction linac can be obtained by modifying previous results developed for circular machines.<sup>5</sup> Since one is "below transition" or in the positive mass regime, there will be instability only in the presence of resistivity. The  $e$ -folding length, far above threshold, is thus

$$\lambda^{-1} = \frac{R}{Z_0} \left( \frac{4\pi^2 S^2 M_p N r_p}{[1 + 2 \ln(b/a)] ML} \right)^{1/2}, \quad (1)$$

where  $a$  is the beam radius,  $b$  is the pipe radius,  $Z = iX + R$  is the impedance per unit length,  $N/L$  is the ion density,  $r_p$  is the classical proton radius,  $Z_0$  is the free space impedance,  $S = q/e$  is the charge state of ions, and  $M/M_p$  is the ion mass in units of proton mass. Letting  $R = 200 \Omega/m$ ,  $S = 2$ ,  $M_p/M = 1/200$ ,  $N/L = 10^{15}/20 \text{ m}$ , and  $b/a = 1.5$  we find a growth length  $\lambda = 300 \text{ m}$ . Since envisioned linac drivers have lengths of kilometers, and because there are strict limits on the beam emittance in order to have a beam hit a small target,  $\lambda$  is uncomfortably short.

A similar calculation may be carried out for the rf linac-storage ring system approach to heavy-ion fusion.<sup>6</sup> One finds that the growth time is comparable to, or shorter than, the storage time so that in this approach, also, there is difficulty created by the longitudinal resistive instability.

The validity of this approach; namely the use of the *unbunched* beam growth rate for a bunched beam is supported by experimental observations on many storage rings as well as by a theoretical analysis by Wang and Pellegrini.<sup>7</sup>

On the other hand Kim<sup>8</sup> has found stability, in the presence of resistivity, of a finite bunch. His model assumed uni-

form density, a step function in momentum space, and the impedance of a uniform structure. We have generalized Kim's analysis to include arbitrary symmetric bunches and general impedances.<sup>9</sup> We found—in first-order perturbation theory—that bunches are stable in the limit of small resistivity compared to reactance, assuming no thermal spread and that disturbances move (in the beam frame) with velocities much less than the beam velocity.

We interpret this stability as arising from the growth of the perturbation as it travels backwards in the beam combined with its subsequent decay after reflection at the bunch end. This contrasts with the picture in an unbunched beam where a perturbation moving backwards never reflects and consequently grows indefinitely (or at least until the linear theory becomes invalid). Thus the bunch end—or the position near it, where waves are reflected—becomes very important in the creation of stability in a bunched beam.

However, the bunch end is exactly the place where the linear theory may break down for its is exactly the place where the unperturbed density is approaching zero. Could the result of stability be an artifact of the linearization of the problem?

Furthermore, the stability which we find in first-order perturbation theory must be reconciled with the instability found by Wang and Pellegrini. Could higher-order terms in the perturbation theory or even lack of convergence of the perturbation theory be the source of this reconciliation?

In Sec. II we derive the basic equation which is used subsequently to analyze the longitudinal oscillations of a bunch. We first (Sec. IIA) derive the equation by taking moments of the Vlasov equation (a hydrodynamic approach) and then (Sec. IIB) by a more careful (but also more complicated) analysis.

In Sec. III we give a careful, and complete, derivation of the result of first-order perturbation theory. The result of this analysis has been reported previously.<sup>9</sup>

In Sec. IV we consider second-order perturbation theory and derive an equation including the effects of the energy

spread of the particle distribution function. In addition we consider the effect of synchrotron motion.

In Sec. V we consider the validity of perturbation theory. In particular, we show that a model can be exhibited for which the linearization of the problem is valid. For this model we have linear reflection of waves at the bunch end, with the result that a nonlinear analysis (which is beyond the scope of this paper) is not required. Within this model, we then study the validity of a perturbation theory analysis. A criterion is obtained that, when violated, leads to essentially unstable motion. We derive an estimate of the growth rate which turns out to be that of Eq. (1).

Since the growth rate of Eq. (1) is unacceptable for heavy-ion fusion schemes the criterion becomes a design criterion. This criterion can, in practice, be met.

We have been motivated by the requirements of heavy-ion fusion—and limited ourselves as a result—to the longitudinal density oscillations of intense particle bunches. In the course of this study, we have developed much understanding of oscillations in finite bunches and how such oscillations “go over” into the (previously studied) oscillations of a very long (infinitely long) beam. In short, we have developed a proper theoretical framework for analysis of oscillations in bunched beams. A similar study can be made—and should be made—of transverse oscillations in bunches and, also, of the coupled motion which results from combining transverse and longitudinal oscillations. These analyses would have application to accelerators that have very intense bunches (such as are presently considered for material studies or free electron laser use).

## II. DERIVATION OF BASIC EQUATION

### A. Hydrodynamic analysis

The equation governing the motion of the ions is the Vlasov equation in the beam frame:

$$\frac{\partial f(x,v,t)}{\partial t} + v \frac{\partial f(x,v,t)}{\partial x} + \frac{qE}{M} \frac{\partial f(x,v,t)}{\partial v} = 0. \quad (2)$$

In writing Eq. (2) the following approximations have been made: (1) transverse motion is decoupled from the longitudinal motion, (2) the beam velocity  $v_B$  is nonrelativistic, and (3) the ions are collisionless.

In Eq. (2),

$$E = E(x,t) = E_A(x,t) + E_S(x,t),$$

where  $E_A(x,t)$  is the applied electric field (which is responsible for the bunching) and  $E_S(x,t)$  is the space charge electric field.

The line density, longitudinal current, and normalization, respectively, are defined as follows:

$$n(x,t) = \int f(x,v,t) dv,$$

$$I(x,t) = \int v f(x,v,t) dv,$$

$$N = \int n(x,t) dx.$$

Other constants are defined as in Sec. I.

We linearize about a steady-state solution  $f_0(x,v)$  satisfies

$$v \frac{\partial f_0(x,v)}{\partial x} + \frac{q}{M} (E_A + E_S^0) \frac{\partial f_0(x,v)}{\partial v} = 0, \quad (3)$$

where  $E_S^0(x)$  is the field due to the equilibrium distribution

$$n_0(x) = \int f_0(x,v) dv.$$

Substituting  $f(x,v,t) = f_0(x,v) + f_1(x,v,t)$  in (2), and using (3) we find

$$\frac{\partial f_1}{\partial t} + v \frac{\partial f_1}{\partial x} + \frac{q}{M} (E_A + E_S^0) \frac{\partial f_1}{\partial v} + \frac{q}{M} E_S^1 \frac{\partial f_0}{\partial v} = 0, \quad (4)$$

where

$$E_S^1 = E_S - E_S^0.$$

Taking moments of Eq. (4) we find

$$\frac{\partial n_1}{\partial t} + \frac{\partial I_1}{\partial x} = 0, \quad (5)$$

and

$$\frac{\partial I_1(x,t)}{\partial t} + \frac{\partial}{\partial x} \int v^2 f_1(x,v,t) dv - \frac{q}{M} (E_A + E_S^0) n_1(x,t) - (q/M) E_S^1 n_0(x) = 0. \quad (6)$$

We can estimate  $\int v^2 f_1(x,v,t) dv$ . Limitations at the target imply that  $\Delta v/v_B < 0.01$  where  $\Delta v$  is the thermal spread in the beam, so  $\partial/\partial x \int v^2 f_1 \sim \partial n_1/\partial x (\Delta v)^2 \sim k n_1 (\Delta v)^2$  where  $k$  is a wavenumber characteristic of the perturbation. Also  $\Delta v$  is related to the synchrotron frequency and the zeroth order distribution via the relation

$$(q/M)(E_A + E_S^0) \sim (\Delta v)^2/L \sim \Omega^2 L,$$

where the synchrotron frequency is  $\Omega$ . Thus for an applied field  $E_A$  carefully matched with  $E_S$ , the synchrotron motion and thermal spread can be neglected. Equation (6) then becomes

$$\frac{\partial I_1}{\partial t} - \frac{q}{M} E_S^1 n_0 = 0. \quad (7)$$

Equations (5) and (7) can be combined to give

$$\frac{\partial^2 n_1(x,t)}{\partial t^2} + \frac{q}{M} \frac{\partial}{\partial x} (E_S^1 n_0) = 0. \quad (8)$$

Really  $E_S$  is a function of current, not density, and is determined via the solution of the Maxwell equations with boundary conditions determined by the geometry and electromagnetic properties of the accelerator.<sup>5</sup> The boundary conditions are known in the lab frame where

$$\tilde{E}(k^l, \omega^l) = -Z(k^l, \omega^l) \tilde{I}^l(k^l, \omega^l).$$

The superscript  $l$  denotes the lab coordinates and fields. For nonrelativistic bunches,  $k^l = k$ ,  $\omega^l = kv_B + \omega$ , and the tilde denotes a Fourier transform:

$$\tilde{G}(k, \omega) = \iint dx dt e^{i(kx - \omega t)} G(x, t).$$

If we restrict to the case  $\omega/k \ll v_B$ , which is good approximation near the center of the bunch (where the beam is ap-

proximately uniform), then the unbunched beam result can be used:

$$\omega/k \cong \{(q^2 N / ML) [1 + 2 \ln(b/a)]\}^{1/2} \sim 0.006 v_B.$$

Thus we approximate  $I^l(k, \omega^l) = v_B n(k, \omega)$ , and since the value of  $E_{\text{long}}$  is independent of beam or lab frame,

$$\tilde{E}_S^l(k, \omega) = -Z(k) v_B \tilde{n}_1(k, \omega), \quad (9)$$

where

$$Z(k) \equiv Z(k, k v_B).$$

Thus Eq. (8) becomes

$$\frac{\partial^2}{\partial t^2} n_1 + \frac{q}{M} \frac{\partial}{\partial x} \left( n_0 \frac{1}{(2\pi)^2} \iint dk d\omega e^{-ikx + i\omega t} \right. \\ \left. \times [-Z(k) v_B \tilde{n}_1(k, \omega)] \right) = 0. \quad (10)$$

This equation can be transformed into an integral equation using the convolution theorem: If  $n(k) = (G * F)(k) \equiv \int G(k - k') F(k') dk'$ , then  $n(x) = 2\pi G(x) F(x)$ . Thus taking the Fourier transform of (10) gives

$$-\omega^2 \tilde{n}_1(k) + \frac{v_B i k q}{2\pi M} \int dk' \tilde{n}_0(k - k') Z(k') \tilde{n}_1(k') = 0. \quad (11)$$

## B. Vlasov analysis

### 1. Motivation

We now present a more general formulation of the beam stability problem starting from the Vlasov equation. By doing this we are able to provide a careful and systematic derivation of the fluid equations used previously, and also to investigate corrections due to thermal effects and to bounce motion.

We continue to assume that perturbed quantities such as the electric field and distribution function vary significantly only in the longitudinal direction, the transverse variation of the electric field outside the beam being included in the effective coupling impedance. We also assume that the boundary conditions are uniform, both temporarily and in the longitudinal direction. In other words, we are interested in frequencies and growth rates small compared to the frequencies of rapid fluctuations (as seen in the beam frame) of the external structures, and thus can replace the actual boundary conditions by those of a time-independent spatially uniform (in the longitudinal direction) wall characterized by an impedance function. We also assume, for simplicity, that the particle motion is nonrelativistic.

It is necessary for us to carefully distinguish lab frame quantities from beam frame quantities. The relationship between electric field and current is known, in terms of impedances, in the laboratory frame of reference. However, since the bunch approaches and completely passes any fixed longitudinal position in the lab, we cannot expect a simple exponential dependence on time of any perturbed quantities; i.e., a dispersion relation doesn't exist in the lab frame. However, a simple exponential dependence on time of the perturbed quantities can be expected in the beam frame if we assume uniform boundary conditions. Thus, the field-current relation must be transformed from the lab frame to the beam frame.

### 2. Formalism

To begin, we write down the linearized Vlasov equation in the lab frame:

$$\frac{\partial f_1^l}{\partial t^l} + v^l \frac{\partial f_1^l}{\partial x^l} + \frac{q E_0}{M} \frac{\partial f_1^l}{\partial v^l} = -\frac{q E_1}{M} \frac{\partial f_0^l}{\partial v^l}. \quad (12)$$

The difficulty in solving this equation is that  $E_0$  and  $f_0$  are both time dependent. The perturbed electric field is a functional of the perturbed current  $I^l$  defined to be

$$I^l(x^l, t^l) = \int dv^l v^l f_1^l(x^l, v^l, t^l). \quad (13)$$

We express the relationship between  $E_1$  and  $I$  by means of the general equation

$$E_1(x^l, t^l) = \int dt'^l \int dx'^l G(t^l - t'^l, x^l - x'^l) I^l(x'^l, t'^l), \quad (14)$$

where causality requires

$$G^l(t^l - t'^l, x^l - x'^l) = 0 \quad \text{if } t^l > t'^l. \quad (15)$$

The fact that the arguments of  $G^l$  are difference variables only is due to our assumption of uniform boundary conditions.

In order to convert Eq. (12) into an integral equation, we define some subsidiary functions. We first define the unperturbed orbit functions as the solutions of

$$\frac{dx_{\text{ORB}}}{dt'^l}(t'^l; x_0^l, v_0^l, t_0^l) = v_{\text{ORB}}(t'^l; x_0^l, v_0^l, t_0^l), \quad (16)$$

$$\frac{dv_{\text{ORB}}}{dt'^l}(t'^l; x_0^l, v_0^l, t_0^l) = \frac{q E_0}{M}(x_{\text{ORB}}, t'^l), \quad (17)$$

which satisfy

$$x_{\text{ORB}}(t_0^l; x_0^l, v_0^l, t_0^l) = x_0^l, \quad (18)$$

$$v_{\text{ORB}}(t_0^l; x_0^l, v_0^l, t_0^l) = v_0^l. \quad (19)$$

We now define a function of six variables:

$$K^l(x^l, v^l, t^l; x_0^l, v_0^l, t_0^l) \\ = \begin{cases} \delta(x^l - x_{\text{ORB}}) \delta(v^l - v_{\text{ORB}}), & t_0^l < t^l, \\ 0 & t_0^l > t^l. \end{cases}$$

The solution of Eq. (12) can now be written as

$$f_1^l(x^l, v^l, t^l) = -\frac{q}{M} \int dt_0^l \int dx_0^l \int dv_0^l \\ \times K(x^l, v^l, t^l; x_0^l, v_0^l, t_0^l) E_1(x_0^l, t_0^l) \\ \times \frac{\partial f_0^l(x_0^l, v_0^l, t_0^l)}{\partial v_0^l}, \quad (20)$$

where  $E_1$  is given by Eq. (14).

Multiplying Eq. (20) by  $v^l$  and integrating to form the current we get

$$I^l(x^l, t^l) = -\frac{q}{M} \int dt_0^l \int dx_0^l \int dv_0^l \int dv^l v^l \\ \times K(x^l, v^l, t^l; x_0^l, v_0^l, t_0^l) \\ \times E_1(x_0^l, t_0^l) \frac{\partial f_0^l(x_0^l, v_0^l, t_0^l)}{\partial v_0^l}, \quad (21)$$

where  $E_1$  is given by Eq. (14).

We now transform to the beam frame of reference by means of

$$\begin{aligned} t' &= t, & x' &= x + v_B t, & v' &= v_B + V, \\ t'_0 &= t_0, & x'_0 &= x_0 + v_B t_0, & v'_0 &= v_B + v_0, \end{aligned} \quad (22)$$

and we define

$$I(x, t) = I'(x + v_B t, t). \quad (23)$$

Note that  $I$  is the lab frame current expressed in beam frame variables, it is not the current seen in the beam frame. Equation (21) now takes the form

$$\begin{aligned} I(x, t) &= -\frac{q}{M} \int dt_0 \int dx_0 \int dv_0 \int dv(v_B + v) \\ &\quad \times \tilde{K}(x + v_B t, v_B + v, t; x_0 + v_B t_0, v_B + v_0, t_0) \\ &\quad \times E_1(x_0 + v_B t_0, t_0) \frac{\partial f'_0}{\partial v_0}(x_0 + v_B t_0, v_B + v_0, t_0). \end{aligned} \quad (24)$$

If we now assume  $E_0 = E_0(x - v_B t)$ , the equilibrium is time independent in the beam frame and the beam frame equilibrium forces are independent of time. We can thus define

$$f_0(x_0, v_0) = f'_0(x_0 + v_B t_0, v_B + v_0, t_0), \quad (25)$$

$$\begin{aligned} K(x, v; x_0, v_0, t - t_0) \\ = K(x + v_B t, v_B + v, t; x_0 + v_B t_0, v_B + v_0, t_0). \end{aligned} \quad (26)$$

The integral in Eq. (24) is a convolution in time, and we can Fourier transform to obtain

$$\begin{aligned} \tilde{I}(x, \omega) &= -\frac{q}{M} \int dx_0 \int dv_0 \int dv(v_B + v) \\ &\quad \times \tilde{K}(x, v; x_0, v_0, \omega) \tilde{E}_1(x_0, \omega) \frac{\partial f_0}{\partial v_0}(x_0, v_0), \end{aligned} \quad (27)$$

where  $\tilde{E}_1(x_0, \omega)$  is the temporal Fourier transform of  $E_1(x_0 + v_B t_0, t_0)$ .

Now, we can use Eq. (14) to obtain

$$\begin{aligned} E_1(x_0 + v_B t_0, t_0) &= \int dt' \int dx' \\ &\quad \times G(t_0 - t', x_0 + v_B t_0 - x') \\ &\quad \times I'(x', t'). \end{aligned} \quad (28)$$

Transforming integration variables and Fourier transforming in time we get

$$E_1(x_0, \omega) = \int dx' G(\omega, x_0 - x') \tilde{I}(x', \omega), \quad (29)$$

where

$$G(\omega, x_0 - x') = \int dt G'(t, x_0 - x' + v_B t) e^{+i\omega t}. \quad (30)$$

Note that in Fourier transforming  $G$  the time dependence in both arguments must be taken into account. We finally arrive at an integral equation for  $I$ ,

$$\tilde{I}(x, \omega) = \int dx' P(x, x', \omega) \tilde{I}(x', \omega), \quad (31)$$

where we have defined

$$\begin{aligned} P(x, x', \omega) &= -\frac{q}{M} \int dx_0 \int dv_0 \int dv(v_B + v) \\ &\quad \times \tilde{K}(x, v; x_0, v_0, \omega) \frac{\partial f_0}{\partial v_0}(x_0, v_0) G(\omega, x_0 - x'). \end{aligned} \quad (32)$$

Let us now consider the cold beam limit, which is obtained by taking

$$F_0(x_0, v_0) = n_0(x_0) \delta(v_0), \quad (33)$$

and

$$K(x, v; x_0, v_0, t) = \begin{cases} \delta(v - v_0) \delta(x - x_0 - v_0 t), & t > 0, \\ 0, & t < 0. \end{cases} \quad (34)$$

We then find that

$$\begin{aligned} P(x, x', \omega) &= \frac{q}{M} \left( \frac{in_0(x) G(\omega, x - x')}{\omega} \right. \\ &\quad \left. + \frac{v_B}{\omega^2} \frac{\partial}{\partial x} [n_0(x) G(\omega, x - x')] \right). \end{aligned} \quad (35)$$

We can now write Eq. (31) as

$$\begin{aligned} \tilde{I}(x, \omega) &= \frac{iq}{M\omega} \left( 1 - \frac{iv_B}{\omega} \frac{\partial}{\partial x} \right) \\ &\quad \times \left( n_0(x) \int dx' G(\omega, x - x') \tilde{I}(x', \omega) \right). \end{aligned} \quad (36)$$

Since we can identify the spatial transform of the Green's function  $G(\omega, x)$  with the impedance  $Z(\omega, k)$ , Eq. (36) is equivalent to Eq. (11) in the limit  $\omega \ll kv_B$ .

### III. FIRST-ORDER PERTURBATION THEORY

#### A. Theorems

An analysis of Eq. (11) leads to some general conclusions about  $\omega$ .

*Case 1.*  $Z(k) = iX(k)$ , where  $X(k) = X^*(k) = -X(-k)$  and  $X(k) < 0$  for  $k > 0$ . Multiplying Eq. (11) by  $-iX(k) \tilde{n}_1^*(k)/k$  and integrating over  $k$  gives

$$\begin{aligned} +i\omega^2 \int dk |\tilde{n}_1(k)|^2 \frac{X(k)}{k} + \frac{iqv_B}{2\pi M} \\ \times \int \int dk' dk \tilde{n}_0(k - k') X(k) X(k') \tilde{n}_1^*(k) \tilde{n}_1(k') = 0, \end{aligned}$$

so

$$\begin{aligned} \omega^2 &= -\frac{qv_B}{2\pi M} \\ &\quad \times \frac{\int \int dk dk' \tilde{n}_0(k - k') X(k) X(k') \tilde{n}_1^*(k) \tilde{n}_1(k')}{\int dk |\tilde{n}_1(k)|^2 X(k)/k}. \end{aligned} \quad (37)$$

The numerator in Eq. (37) is of the form

$$\int dk \tilde{F}(k) \tilde{G}^*(k),$$

where

$$\tilde{F}(k) = \int dk' \tilde{n}_0(k - k') X(k') \tilde{n}_1(k),$$

and

$$\tilde{G}(k) = X(k) \tilde{n}_1(k).$$

Then Plancherel's theorem,

$$\int dk \tilde{F}(k) \tilde{G}^*(k) = 2\pi \int F(x) G^*(x) dx,$$

gives

$$\begin{aligned} & \int \tilde{n}_0(k - k') X(k') \tilde{n}_1(k') X(k) \tilde{n}_1^*(k) dk dk' \\ & = (2\pi)^2 \int dx n_0(x) |E_S(x)|^2 > 0. \end{aligned}$$

Since  $X(k)$  is odd and  $X(k) < 0$  for  $k > 0$ ,

$$\int dk |\tilde{n}_1(k)|^2 \frac{X(k)}{k} < 0.$$

Therefore  $\omega^2 > 0$  and  $\omega$  is real. The assumption that  $X(k)$  is negative for  $k > 0$  simply states that we are not in the "negative mass regime."

Having established stability for a reactive impedance,

we now consider an impedance with a small resistive part.

Take

$$Z(k) = iX(k) + R(k), \quad R(k) = R^*(k) = R(-k),$$

and assume that Eq. (11) has been solved for  $Z(k) = iX(k)$  yielding a set of eigenfunctions  $n_n^{(0)}$  satisfying

$$\begin{aligned} & -(\omega_n^{(0)})^2 \tilde{n}_n^{(0)} + ik \frac{qv_B}{2\pi M} \\ & \times \int \tilde{n}_0(k - k') iX(k') \tilde{n}_n^{(0)}(k') dk' = 0. \end{aligned} \quad (38)$$

Additionally, assume the eigenfrequencies  $\omega_n$  are nondegenerate. For  $R \ll X$ , expand  $\tilde{n} = \tilde{n}_n^{(0)} + \epsilon \tilde{n}_n^{(1)}$ , and  $\omega = \omega_n^{(0)} + \epsilon \omega_n^{(1)}$  in Eq. (11) so

$$\begin{aligned} & -2\omega_n^{(0)} \omega_n^{(1)} \tilde{n}_n^{(0)} - (\omega_n^{(0)})^2 \tilde{n}_n^{(1)} + \frac{qv_B}{2\pi M} ik \int \tilde{n}_0(k - k') \\ & \times [iX(k') \tilde{n}_n^{(1)}(k') + R(k') \tilde{n}_n^{(0)}(k')] dk' = 0. \end{aligned}$$

Multiplying by  $-in_n^{(0)*}(k)X(k)/k$  and integrating over  $k$  yields

$$\begin{aligned} & i\omega_n^{(0)} \omega_n^{(1)} \int |\tilde{n}_n^{(0)}(k)|^2 \frac{X(k)}{k} dk + \frac{qv_B}{2\pi M} \iint \tilde{n}_0(k - k') R(k') X(k) \tilde{n}_n^{(0)}(k') \tilde{n}_n^{(0)*}(k) dk dk' \\ & + i(\omega_n^{(0)})^2 \int \frac{X(k)}{k} \tilde{n}_n^{(0)*}(k) \tilde{n}_n^{(1)}(k) dk + \frac{iqv_B}{2\pi M} \iint \tilde{n}_0(k - k') X(k) X(k') \tilde{n}_n^{(0)*}(k) \tilde{n}_n^{(0)}(k') = 0. \end{aligned} \quad (39)$$

Since  $n_0(x)$  is real and symmetric, so is its Fourier transform  $\tilde{n}_0(k)$ . The variables  $(k, k')$  may be interchanged in the last term above, so the last two terms of Eq. (39) can be grouped together. Using Eq. (38) we find

$$\int dk \frac{-iX(k) \tilde{n}_n^{(1)}(k)}{k} \left( -(\omega_n^{(0)})^2 \tilde{n}_n^{(0)}(k) + \frac{qv_B ik}{2\pi M} \int dk' \tilde{n}_0(k - k') iX(k') \tilde{n}_n^{(0)}(k') \right) = 0.$$

Equation (39) can now be solved for  $\omega^{(1)}$ :

$$\omega_n^{(1)} = \frac{iqv_B}{2\pi M} \frac{\iint dk dk' \tilde{n}_0(k - k') X(k) R(k') \tilde{n}_n^{(0)}(k') \tilde{n}_n^{(0)*}(k)}{2\omega_n^{(0)} \int |\tilde{n}_n^{(0)}(k)|^2 X(k)/k}. \quad (40)$$

The numerator of Eq. (40) is proportional to

$$\iint dk dk' \tilde{n}_0(k - k') X(k) R(k') \tilde{n}_n^{(0)*}(k) \tilde{n}_n^{(0)}(k') = \int dk \tilde{F}(k) \tilde{G}^*(k),$$

where

$$\tilde{F}(k) = \int dk' \tilde{n}_0(k - k') \tilde{n}_n^{(0)}(k') R(k') \quad \text{and} \quad \tilde{G}(k) = X(k) \tilde{n}_n^{(0)*}(k).$$

Furthermore note that since  $X(k)$  is an odd function of  $k$

$$\begin{aligned} & -(\omega_n^{(0)})^2 \tilde{n}_n^{(0)}(-k) + \frac{iqkv_B}{2\pi M} \int \tilde{n}_0(k - k') iX(k') \tilde{n}_n^{(0)}(-k') dk' \\ & = -(\omega_n^{(0)})^2 \tilde{n}_n^{(0)}(-k) + \frac{iqkv_B}{2\pi M} \int \tilde{n}_0(k - k') iX(-k') \tilde{n}_n^{(0)}(-k') dk' \\ & = -(\omega_n^{(0)})^2 \tilde{n}_n^{(0)}(-k) + (-k) \frac{iqv_B}{2\pi M} \int \tilde{n}_0(-k - k') iX(k') \tilde{n}_n^{(0)}(k') dk' = 0, \end{aligned}$$

since this is Eq. (11) with  $k \rightarrow -k$ .

Thus for nondegenerate eigenvalues, the eigenfunctions must have definite parity. This implies  $\tilde{F}(k)$  has the same parity as  $\tilde{n}_n(k)$  and  $\tilde{G}(k)$  has the opposite parity of  $\tilde{n}_n(k)$ ,

whence

$$\int dk \tilde{F}(k) \tilde{G}(k) = 0.$$

We thus find

$$\omega_n^{(1)} = 0.$$

One should note that this analysis breaks down for asymmetric  $n_0(x)$ .

We should have shown that quite generally bunches are stable under small resistive perturbations. Before examining in detail the validity of the approximations made, we derive some well-known results.

## B. Special cases

For a wavelength that is long compared to the beam and accelerator transverse dimensions, but short compared to a bunch length, the impedance can be taken as<sup>5</sup>

$$Z(k) = -iqk(g_0/v_B) - qR', \quad (41)$$

where  $g_0$  is the usual geometrical factor  $g_0 = 1 + 2 \ln(b/a)$  for cylindrical geometry. Equation (8) then becomes

$$\frac{\partial^2 n_1(x,t)}{\partial t^2} - \frac{q^2}{M} \frac{\partial}{\partial x} \times \left[ \left( g_0 \frac{\partial n_1(x,t)}{\partial x} + v_B R' n_1(x,t) \right) n_0(x) \right] = 0. \quad (42)$$

We further specialize:

*Case 1.* Here  $R' = 0$ ,  $n_0(x) = n_0 = \text{const}$ . Fourier transforming gives the dispersion relation

$$-\omega^2 + \frac{q^2 n_0 g_0}{M} k^2 = 0 \quad \text{or} \quad \frac{\omega}{k} = \left( \frac{q^2 g_0 n_0}{M} \right)^{1/2} = v_p;$$

as quoted earlier [above Eq. (9)], this is the old result for an unbunched beam.

*Case 2.* Here  $n_0(x) = n_0$ ,  $R' \neq 0$  then

$$-\omega^2 + \frac{q^2 g_0 n_0}{M} k^2 + ik \frac{q^2 v_B R' n_0}{M} = 0,$$

whence

$$\omega = k \left[ \frac{q^2 g_0 n_0}{M} \left( 1 + \frac{iv_B R'}{g_0 k} \right) \right]^{1/2} \cong kv_p + \frac{iR' v_B v_p}{2g_0},$$

for  $R' \ll kg_0/v_B$ . Thus the unbunched beam has a growth rate  $\text{Im}(\omega)$  given by

$$\text{Im}(\omega) = R' v_B v_p / 2g_0, \quad (43)$$

which is exactly the result quoted in Eq. (1).

The result (43) corresponds to an  $e$ -folding distance  $\lambda$  given by

$$\lambda^{-1} = \text{Im}(\omega)/v_B \cong R' v_p / g_0.$$

*Case 3.* We now consider the physically more realistic parabolic line density

$$n_0(x) = n_0(0)(1 - x^2/L^2),$$

$n_0(0) = 3N/4L$  (normalization), and assume  $R' = 0$ . Then Eq. (42) becomes

$$\omega^2 n_1 + \frac{q^2 g_0 n_0(0)}{M} \frac{\partial}{\partial x} \left( 1 - \frac{x^2}{L^2} \right) \frac{\partial n_1}{\partial x} = 0. \quad (44)$$

Equation (44) is Legendre's equation, with

$$n_1(x) = P_m(x/L), \quad \omega_m = (v_p/L)[m(m+1)]^{1/2}, \quad (45)$$

where  $m = 0, 1, 2$  and  $v_p^2 = q^2 g_0 n_0(0)/M$ . Since  $[m(m+1)]^{1/2}/L \sim k$ , again  $\omega/k \sim v_p$ . This is the result previously obtained by Neuffer.<sup>10</sup>

In the above examples we have considered impedances whose imaginary part grows linearly with wavenumber  $k$ . Those impedances are good for wavelengths long compared to the beam radius  $r_b$ . A better phenomenological impedance, including the effect of finite pipe radius, is

$$X(k) = \frac{-q(g_0/v_B)k}{1 + (dk)^2}, \quad (46)$$

where  $d^2 = r_b^2 g_0/4$ . In the limit  $k \gg d^{-1}$ , Eq. (46) gives the relation between  $E$  and  $n$  expected from Coulomb's law.

*Case 4.* Here  $n_0(x) = n_0$ ,  $R' = 0$ ,  $X$  is given by Eq. (47). We find the unbunched beam dispersion relation to be

$$\omega^2 = (q^2 g_0 n_0 / M) [k^2 / (1 + k^2 d^2)]. \quad (47)$$

Thus we see that coasting beams are linearly unstable, whereas bunched beams have temporal stability under small resistive perturbations. Our physical picture is that the wave decays as it moves forward, reflects near the end of the bunch, and then grows as it moves backwards. Thus, if the modes are nondegenerate, there is growth followed by decay followed by growth, but no net instability.

## C. Transient spatial amplification

In any resistive structure there will be transient spatial amplification even in the eigenfrequencies are real. In this section we will estimate the spatial growth of a perturbation in the middle of a bunch, where we assume a slowly varying unperturbed density.

We consider the usual impedance  $Z(k) = -iqk(g_0/v_B) - R'q$ , so that the perturbed density is governed by Eq. (42). Taking the Fourier transform in time of Eq. (42) yields

$$\frac{\partial^2 n_1(x,\omega)}{\partial x^2} + \left( \frac{1}{n_0} \frac{dn_0}{dx} + \frac{v_B R'}{g_0} \right) \frac{\partial n_1(x,\omega)}{\partial x} + \left( \frac{\omega^2}{v_p^2} + \frac{R' v_B}{g_0 n_0} \frac{dn_0}{dx} \right) n_1(x,\omega) = 0. \quad (48)$$

Setting

$$\Psi(x,\omega) = n_0^{1/2}(x) e^{v_B R' x / 2g_0} n_1(x,\omega), \quad (49)$$

and using Eq. (48), we find  $\Psi(x,\omega)$  satisfies

$$\frac{\partial^2 \Psi}{\partial x^2} + \left[ \frac{\omega^2}{v_p^2} + \frac{R' v_B}{n_0 g_0} \frac{dn_0}{dx} - \frac{1}{4} \left( \frac{dn_0/dx}{n_0} + \frac{R' v_B}{g_0} \right)^2 - \frac{1}{2} \frac{d}{dx} \left( \frac{dn_0/dx}{n_0} \right) \right] \Psi = 0. \quad (50)$$

For a slowly varying  $n_0(x)$  Eq. (50) simplifies to

$$\frac{\partial^2 \Psi}{\partial x^2} + \left[ \frac{\omega^2}{v_p^2} - \left( \frac{R' v_B}{2g_0} \right)^2 \right] \Psi = 0. \quad (51)$$

Explicitly we have assumed that

$$\frac{R' v_B}{g_0 n_0} \frac{dn_0}{dx}, \quad \frac{d}{dx} \left( \frac{dn_0/dx}{n_0} \right), \quad \left( \frac{dn_0/dx}{n_0} \right)^2 \ll \frac{\omega^2}{v_p^2}, \quad \left( \frac{R' v_B}{g_0} \right)^2.$$

Equation (51) has solutions of the form

$$\Psi(x, \omega) = k^{-1/2}(x) \exp\left(+i \int k(x') dx'\right), \quad (52)$$

where

$$k(x) = \left[ \frac{\omega^2}{v_p^2} - \left( \frac{R' V_B}{2g_0} \right)^2 \right]^{1/2}. \quad (53)$$

Thus  $k(x)$  is shifted as the resistivity increases, and  $k(x)$  vanishes when the unbunched beam growth rate equals the oscillation frequency  $\omega$ .

Using Eq. (49) the magnitude of the perturbed density is

$$n_1(x, t) = e^{-xR'v_B/2g_0} [n_0(x)(1 - \alpha^2)]^{1/4} \quad (54)$$

where  $\alpha = R'v_Bv_p/2g_0$ . The first term,  $\exp(-xv_B R'/2)$ , is the spatial growth (decay) of the backward (forward) moving wave in an unbunched beam. It can easily be derived by assuming  $\omega$  real and  $k$  complex in Sec. IIIB, Case 2. The second term in Eq. (54) represents a modification of this unbunched beam result arising from slow variation in the unperturbed density. Since we are considering small resistivities,  $\alpha < 1$ ; furthermore  $[n_0(x)]^{-1/4}$  increases very slowly, and only becomes appreciable at the bunch ends where we must examine our approximations in greater detail (see Sec. V). In short there is little modification of the unbunched beam spatial amplification within regions of slowly varying density.

## IV. SECOND-ORDER PERTURBATION THEORY

### A. Damping rate

Since our first-order perturbation theory yielded neither growth nor damping, a second-order calculation is needed to see if resistive effects are stabilizing or destabilizing. We employ the formalism developed in Sec. IIB and perform an expansion in both  $\omega/kv_B$  and  $R'$ . First- and second-order perturbation theories then give the damping rate for the parabolic bunch. We assume the standard model<sup>5</sup> for the impedance which corresponds to

$$G(t, x) = G_1(t) \delta(x) + G_2(t) \delta''(x). \quad (55)$$

The Fourier transforms of  $G_1$  and  $G_2$  are the resistive and reactive parts of the impedance, denoted by  $Z_1(\omega)$  and  $Z_2(\omega)$ , respectively. Using Eq. (55) in the definition (30) we find

$$G(\omega, x - x') = \int dt e^{+i\omega t} G_1(t) \delta''(x + v_B t - x') + \int dt e^{i\omega t} G_2(t) \delta''(x + v_B t - x'). \quad (56)$$

In order to simplify this expression we use the formal identity for arbitrary functions  $f$ ,

$$f(x + v_B t) = \exp\left(v_B t \frac{\partial}{\partial x}\right) f(x). \quad (57)$$

Note that our formal manipulations can be rigorously justified if we restrict our considerations to  $C$  perturbations of compact support,<sup>11</sup> a restriction we must apply in any case to use the  $\delta$  functions in Eq. (55). Using Eq. (57) we see that Eq. (56) becomes

$$G(\omega, x - x') = Z_1\left(\omega - iv_B \frac{\partial}{\partial x}\right) \delta(x - x') + Z_2\left(\omega - iv_B \frac{\partial}{\partial x}\right) \delta''(x - x'). \quad (58)$$

Roughly speaking, the effect of the transformation of frames has been to replace the beam frame frequency  $\omega$  by the equivalent lab frame frequency  $\omega + kv_B$ , where  $k$  is the wavenumber. Equation (36) can now be written, in this case, as

$$\tilde{I}(x, \omega) = \frac{iq}{M\omega} \left(1 - \frac{iv_B}{\omega} \frac{\partial}{\partial x}\right) n_0(x) \left[ Z_1\left(\omega - iv_B \frac{\partial}{\partial x}\right) \tilde{I}(x, \omega) + Z_2\left(\omega - iv_B \frac{\partial}{\partial x}\right) \frac{\partial^2 \tilde{I}}{\partial x^2} \right]. \quad (59)$$

For the standard model<sup>5</sup> we have

$$Z_1(\omega) = -R'q, \quad (60)$$

$$Z_2(\omega) = ig_0q/\omega, \quad (61)$$

where  $R'$  is a resistivity. Using these expressions in Eq. (59) we find

$$\tilde{I}(x, \omega) = \frac{iq}{M\omega} \left(1 - \frac{iv_B}{\omega} \frac{\partial}{\partial x}\right) n_0(x) \times \left( -R' \tilde{I}(x, \omega) + \frac{ig_0}{\omega - iv_B(\partial/\partial x)} \frac{\partial^2 \tilde{I}}{\partial x^2} \right). \quad (62)$$

Now, for the case we are interested in

$$|\omega| \ll |kv_B|. \quad (63)$$

So we formally expand the inverse operator as

$$\left(\omega - iv_B \frac{\partial}{\partial x}\right)^{-1} = \left(-iv_B \frac{\partial}{\partial x}\right)^{-1} - \omega \left(-v_B \frac{\partial}{\partial x}\right)^{-2} \quad (64)$$

With this expansion, Eq. (62) becomes

$$\tilde{I}(x, \omega) = \frac{iq}{M\omega} \left(1 - \frac{iv_B}{\omega} \frac{\partial}{\partial x}\right) n_0(x) \times \left[ -R' \tilde{I}(x, \omega) + \frac{ig_0\omega}{V_B^2} \left(1 + \frac{iv_B}{\omega} \frac{\partial}{\partial x}\right) \tilde{I}(x, \omega) \right]. \quad (65)$$

Converting Eq. (65) to the standard form for second-order differential equations, we find

$$\frac{\partial^2 \tilde{I}}{\partial x^2} + \frac{\partial \tilde{I}}{\partial x} \left( \frac{1}{n_0} \frac{dn_0}{dx} + \frac{v_B R'}{g_0} \right) + I \left[ \frac{\omega^2}{v_p^2} + \left( -\frac{\omega R'}{ig_0} + \frac{\omega^2}{v_B^2} \right) \right] \times \left( 1 - \frac{iv_B}{\omega} \frac{1}{n_0} \frac{dn_0}{dx} \right) = 0. \quad (66)$$

We now show that Eq. (66) possesses stable eigenmodes and calculate the damping rates. We use perturbation theory to calculate the damping rate, and thus assume  $R'$  small and  $v_p/v_B \ll 1$ .

We now divide the differential operator into a "large" part

$$M_0 = \frac{\partial^2}{\partial x^2} + \frac{1}{n_0} \frac{dn_0}{dx} \frac{\partial}{\partial x} + \frac{\omega^2}{v_p^2(x)}, \quad (68)$$

and a "small" part

$$\epsilon M = + \frac{v_B R'}{g_0} \frac{\partial}{\partial x} + - \frac{\omega R'}{ig_0} + \frac{\omega^2}{v_B^2} \left( 1 - \frac{iv_B}{\omega} \frac{1}{n_0} \frac{dn_0}{dx} \right). \quad (69)$$

As in Sec. IIB, we follow Neuffer<sup>10</sup> and choose

$$n_0(x) = n_0(0)(1 - x^2/L^2). \quad (70)$$

The eigenfunctions of  $M_0$  are then

$$I_m^{(0)}(x) = P_m(x/L), \quad (71)$$

with  $m = 1, 2, 3, \dots$ , and the eigenvalues are

$$\omega_m^{(0)} = \pm [v_p(0)/L] [m(m+1)]^{1/2}. \quad (72)$$

In order to compute the growth rates to order  $v_p/v_B$  consistently, both first- and second-order terms in the eigenvalue perturbation equation must be retained. Because the operator  $\epsilon M_1$  depends on the eigenvalue, the perturbation theory required is slightly different from the usual, so we present a detailed derivation.

We assume that the eigenfunction and eigenvalue can be expanded as

$$I_n = I_n^{(0)}(x) + \epsilon I_n^{(1)}(x) + \epsilon^2 I_n^{(2)}(x) + \dots, \\ \omega_n = \omega_n^{(0)} + \epsilon \omega_n^{(1)} + \epsilon^2 \omega_n^{(2)} + \dots \quad (73)$$

The operator equation can then be written as

$$\left( M_0(x, \omega_n) + \epsilon \omega_n^{(1)} \frac{\partial M_0}{\partial \omega}(x, \omega_n^{(0)}) \right. \\ \left. + \epsilon^2 \omega_n^{(2)} \frac{\partial M_0}{\partial \omega}(x, \omega_n^{(0)}) + \frac{\epsilon^2 \omega_n^{(1)^2}}{2} \frac{\partial^2 M_0}{\partial \omega^2}(x, \omega_n^{(0)}) \right. \\ \left. + \epsilon M_1(x, \omega_n) + \epsilon^2 \omega_n^{(1)} \frac{\partial M_1}{\partial \omega}(x, \omega_n) + \dots \right) \\ \times (I_n^{(0)} + \epsilon I_n^{(1)} + \epsilon^2 I_n^{(2)} + \dots) = 0. \quad (74)$$

The zeroth-order terms in this equation vanish identically, and the first-order terms yield

$$M_0(x, \omega_n) I_n^{(1)} + \omega_n^{(1)} \frac{\partial M_0}{\partial \omega} I_n^{(0)} + M_1 I_n^{(0)} = 0. \quad (75)$$

Multiplying Eq. (75) by  $I_n^{(0)}$ , integrating over  $x$ , and using the fact that  $M_0$  is self-adjoint we find that the first-order frequency shift is

$$\omega_n^{(1)} = - \frac{\int I_n^{(0)}(x) M_1 I_n^{(0)}(x) dx}{\int I_n^{(0)}(x) (\partial M_0 / \partial \omega) I_n^{(0)}(x) dx}, \quad (76)$$

which is the usual first-order result. To find the second-order frequency shift we need the first-order corrections to the eigenfunction which can be expanded as

$$I_n^{(1)} = A_m I_m^{(0)}(x), \quad n \neq m. \quad (77)$$

Using this expansion in Eq. (75) we find

$$A_m = \frac{-\omega_n^{(1)} \int I_m^{(0)}(x) (\partial M_0 / \partial \omega)(x, \omega_n^{(0)}) I_n^{(0)}(x) - \int I_m^{(0)}(x) M_1 I_n^{(0)}(x)}{\int I_m^{(0)} M_0 I_m^{(0)}}. \quad (78)$$

The second-order terms in Eq. (74) yield

$$M_0 I_n^{(2)} + \omega_n^{(1)} \frac{\partial M_0}{\partial \omega} I_n^{(1)} + M_1 I_n^{(1)} + \omega_n^{(2)} \frac{\partial M_0}{\partial \omega} I_n^{(0)} \\ + \frac{\omega_n^{(1)^2}}{2} \frac{\partial^2 M_0}{\partial \omega^2} I_n^{(0)} + \omega_n^{(1)} \frac{\partial M_1}{\partial \omega} I_n^{(0)} = 0. \quad (79)$$

Multiplying Eq. (79) by  $I_n^{(0)}$ , integrating over  $x$ , and using the self-adjointness of  $M_0$  find

$$\omega_n^{(2)} = - \left( \omega_n^{(1)} \int I_n^{(0)} \frac{\partial M_0}{\partial \omega} I_n^{(1)} + \int I_n^{(0)} M_1 I_n^{(1)} \right. \\ \left. + \frac{\omega_n^{(1)^2}}{2} \int I_n^{(0)} \frac{\partial^2 M_0}{\partial \omega^2} I_n^{(0)} + \omega_n^{(1)} \int I_n^{(0)} \frac{\partial M_1}{\partial \omega} I_n^{(0)} \right) \\ \times \left( \int I_n^{(0)} \frac{\partial M_0}{\partial \omega} I_n^{(0)} \right)^{-1} \quad (80)$$

The  $I_n^{(1)}$  in Eq. (80) are given by Eqs. (77) and (78) which are in turn determined by the operators  $M_0$  and  $M_1$  and by the zeroth-order eigenfunctions, Eq. (71).

The first-order frequency shift is easily calculated and is given by

$$\omega_n^{(1)} = \frac{v_p^2(0)}{2} \left( -iR' - \frac{g_0 \omega_n}{v_B^2} \right) X(n), \quad (81)$$

where the function  $X(n)$  is

$$X(n) = \frac{2n^2 + 2n - 2}{4n^2 + 4n - 3}, \quad (82)$$

and is approximately  $1/2$  for  $n \gg 2$ . We note that in first order both a real frequency shift and a decay rate appear.

A tedious but straightforward calculation of the second-order frequency shift gives

$$\text{Im} \omega_n^{(2)} = [v_p^2(0) R' / g_0] \phi(n), \quad (83)$$

where

$$\phi(n) = - \frac{1}{(2n+1)} \frac{n+1}{2n+3} - \frac{n}{2n-1}. \quad (84)$$

In Eq. (83), we have shown only the contribution to the decay rate and have ignored the additional real frequency shift. We have also ignored various terms of higher order in  $(v_p/v_B)$ . Higher orders of perturbation theory yield no new contributions to the decay rate to this order in  $(v_p/v_B)$ . Note that for  $n > 2$  the second-order contribution can be neglected in comparison to the first-order contribution.

The growth rate for an unbunched beam is

$$\text{Im}(\omega_{\text{unbunched}}) = v_p v_B R' / 2g_0. \quad (85)$$

Thus, we see that

$$\left| \frac{\text{Im}(\omega_{\text{bunched}})}{\text{Im}(\omega_{\text{unbunched}})} \right| = \frac{g_0}{2} \frac{v_p}{v_B} \ll 1. \quad (86)$$

## B. Temperature terms

We indicate briefly how thermal effects can be included if necessary. The distribution function of Eq. (33) was chosen



to have the correct zero-order velocity moment, but has vanishing higher-order velocity moments. To include thermal corrections, we choose a distribution function with nonzero moments through second order. Thus, we now take

$$f_0(x_0, v_0) = n_0(x_0)\delta(v_0) + [T(x_0)/2M]\delta''(v_0), \quad (87)$$

where  $T(x_0)$  is a position dependent "temperature." Using this distribution function instead of Eq. (33) a straightforward calculation yields the equation

$$\begin{aligned} \frac{\partial^2 \tilde{I}}{\partial x^2} + \frac{\partial \tilde{I}}{\partial x} \left( \frac{1}{n_0} \frac{dn_0}{dx} - \frac{v_B R'}{g_0} \right) \\ + \tilde{I} \left[ \frac{\omega^2}{v_p^2} + \left( \frac{\omega R'}{ig_0} + \frac{\omega^2}{v_B^2} \right) \left( 1 - \frac{iv_B}{\omega} \frac{1}{n_0} \frac{dn_0}{dx} \right) \right] \\ = \frac{3}{mv_B^2 n_0} \frac{\partial^2}{\partial x^2} \left[ \left( 1 - \frac{iv_B}{\omega} \frac{\partial}{\partial x} \right) \right. \\ \left. \times n_0 T \left( 1 + \frac{iv_B}{\omega} \frac{\partial}{\partial x} \right) \tilde{I} \right]. \quad (88) \end{aligned}$$

We have written all the additional terms due to the temperature on the right-hand side of this equation. We note that the equation is now a fourth-order differential equation. If we compare the magnitude of the right-hand side to the magnitude of the terms which gave the instability we find

$$\frac{M_1 \text{ temperature}}{M_1 \text{ resistance}} \sim \frac{3g_0 k}{v_B R'} \frac{T}{mv_p^2(0)}, \quad (89)$$

where  $k$  is the effective wavenumber of perturbation. Note that the dependences on  $k, T, R'$ , and  $n_0$  in Eq. (89) are exactly what one would naively expect. For induction linac parameters this ratio is roughly of order 1 for perturbation wavelengths of 25 cm. Thus for long-wavelength disturbances temperatures corrections are unimportant, while for short wavelengths they dominate. To properly treat the temperature corrections on the short-wavelength modes we should use the modified-impedance model, Eq. (46).

The effect of the temperature correction on the short-wavelength modified-impedance model is subtle. Using Eq. (87) again, one can derive the equation

$$\left( \frac{v_p^2(x)}{\omega^2} - d^2 \right) \frac{d^2 \tilde{\phi}}{dx^2} + \tilde{\phi} = \frac{3g_0 q^2}{M\omega^4} \frac{\partial^2}{\partial x^2} n_0 T \frac{\partial^2 \tilde{\phi}}{\partial x^2}, \quad (90)$$

where  $I$  is related to  $\phi$  by

$$\tilde{I} = -\frac{\partial}{\partial x} \left( 1 - d^2 \frac{\partial^2}{\partial x^2} \right) \phi, \quad (91)$$

and  $d^2$  is defined in Sec. IIC, and where the reactive part of the impedance has been replaced by

$$Z(k, \omega) = ig_0 k^2 q / \omega [1 + (dk)^2] \quad (92)$$

in the spatial Fourier transform space. In Eq. (90) we have ignored resistive terms.

The second-order equation [left-hand side of Eq. (90)] has a singularity at  $v_p^2 = (d\omega)^2$ , but the full fourth-order equation is regular there. A careful treatment of this phenomenon requires a singular perturbation theory<sup>12</sup> which we will not present.

As we have pointed out, the impedance-modified temperature-corrected model (for brevity, the "fourth-order"

model), Eq. (90), possess the properties needed for a consistent linearization. Let us examine its dispersion relation, i.e., we take the bunch uniform and assume an  $e^{ikx}$  dependence. We then find

$$-k^2 \left( \frac{v_p^2}{\omega^2} - d^2 \right) + 1 = \frac{3v_p^2(T/m)k^4}{\omega^4}. \quad (93)$$

Solving for  $\omega^2$  we find

$$\begin{aligned} \omega^2 = \frac{k^2 v_p^2}{2(1 + k^2 d^2)} + \frac{1}{2} \left[ \left( \frac{k^2 v_p^2}{1 + k^2 d^2} \right)^2 \right. \\ \left. + \frac{12k^4 v_p^2(T/m)}{1 + k^2 d^2} \right]^{1/2} \quad (94) \end{aligned}$$

If we take the limit where  $kd > 1$  but where

$$12(T/m) < v_p^2 / (kd)^4, \quad (95)$$

then we have approximately

$$\omega^2 = v_p^2 / d^2 + 3k^2(T/M). \quad (96)$$

If we identify  $d^2$  so that

$$v_p^2 / d^2 = \omega_p^2, \quad (97)$$

where  $\omega_p$  is the plasma frequency, then this becomes

$$\omega^2 = \omega_p^2 + 3k^2(T/M), \quad (98)$$

which is the correct dispersion relation for waves in a warm plasma.<sup>13</sup> Thus, our fourth-order model does reduce correctly for short-wavelength modes.

Our treatment of thermal effects is somewhat inconsistent since we have corrected the distribution function but not corrected the orbit functions. The first-order corrections to the orbits are

$$\begin{aligned} x = x_0 + v_0(t - t_0) - \frac{1}{2}x_0\Omega^2(t - t_0)^2 - \frac{1}{2}v_0\Omega^2(t - t_0)^3, \\ v = v_0 - x_0\Omega^2(t - t_0) - \frac{1}{2}v_0\Omega^2(t - t_0)^2, \quad (99) \end{aligned}$$

where  $\Omega$  is the synchrotron frequency. These orbit corrections include the effects of particles bouncing off the ends of the bunch once. Inserting Eqs. (99) into the  $\delta$  functions in Eq. (34), a very tedious but straightforward calculation yields a further correction to  $\epsilon M_1$ :

$$\begin{aligned} (\epsilon M_1)_{\text{bounce}} \\ = -\frac{\Omega^2}{\omega^2 n_0} \frac{\partial}{\partial x} \left[ \left( 3x \frac{dn_0}{dx} - n_0 \right) \left( -\frac{i\omega}{v_B} + \frac{\partial}{\partial x} \right) \right. \\ \left. + \left( \frac{3n_0 i\omega x}{v_B} \frac{\partial}{\partial x} - n_0 \frac{\partial}{\partial x} - 3n_0 x \frac{\partial^2}{\partial x^2} \right) \right]. \quad (100) \end{aligned}$$

We see that these terms contribute to the third-order derivative and lower terms and are of order  $\Omega^2/\omega^2$  compared to the zero-order terms. Note that we have discarded terms involving products of  $\Omega^2$  with  $R'$  and  $T$ , and have discarded higher-order terms in  $(v_p/v_B)$ . Taking the ratio of these terms to the damping term we find

$$\frac{\epsilon M_1 \text{ bounce}}{\epsilon M_1 \text{ resistance}} \sim \frac{3k^2 L g_0}{v_B R'} \left( \frac{\Omega}{\omega} \right)^2. \quad (101)$$

This ratio is of order 1/10, thus, bounce effects are small compared to the decay rate of the parameters relevant to the induction linac.

## V. LIMITS OF VALIDITY

### A. Effective growth

In the previous sections we arrived at the optimistic conclusion that intense bunched beams will be stable to longitudinal perturbations. We must now inquire into the limits of validity of our analysis and the effect of any violations of these limits.

Our analysis has rested on three assumptions: (1) Linearization of the fluid or Vlasov equations; (2) a perturbation theory to calculate the effect of resistivity on the eigenvalues; and (3) nondegeneracy of the eigenmodes of the zero-resistivity equations. Let us comment on each of these assumptions in turn.

Whether or not linearization of the equations is valid depends on a number of circumstances, such as the amplitudes of initial or accidentally applied perturbations and the contributions of nonlinear terms which differ with different density profiles. It is reasonable, however, to proceed with the linearization on the assumption that linear stability criteria are likely indicators of troublesome parameter regimes. Let us note, on the other hand, that the linearization must be *a posteriori* self-consistent, i.e., the modes resulting from the linear analysis must satisfy the assumption that  $n_1 \ll n_0$ . That linearization is not always self-consistent is shown by the parabolic density model Eq. (44), which has modes,  $P_n(x/L)$  and/or  $Q_n(x/L)$  that are nonzero at  $x = \pm L$  where  $n_0 = 0$ . Thus, the results of this and similarly inconsistent models must be seriously questioned.

In fact, it seems to be a property of all low-order models in which the density goes to zero at finite distance that linearity is violated. Roughly, this is due to the fact that the characteristics of the associated partial differential equation "pile up" at zero density and thus perturbations will become arbitrarily large unless the zero density point is at infinity. The "piling up" will also be a problem for spatially infinite models unless the unperturbed density goes to zero slowly enough. Even with slowly decaying density, the standard-impedance model, Eq. (41) with  $R' = 0$ , violates the linearity assumption because perturbation do not spatially decay. The modified-impedance model, Eq. (46), possesses decaying modes near spatial infinity, but is inadequate because the equation possesses singularities at finite  $x$ ; it is generally impossible to require both finiteness at singular points and decay at infinity. On the other hand, the fourth-order model and slowly decaying unperturbed density is consistent with the linearization hypothesis. It has modes with exponential decay at infinity and the temperature has regularized the equation for all finite  $x$ . Thus this is a mathematically well-posed problem once the requirement of decay at infinity is applied.

Our second assumption was that the effect of resistivity could be computed by a perturbation expansion in  $R'/X$ . This assumption has two inherent limitations: (1)  $R'/X$  may not be small, and (2) the expansion may not converge. First,  $R'/X$  may not be small if  $X$  has a zero, i.e., near a resonance. Pellegrini and Wang<sup>7</sup> have investigated such a resonance model and found instability. Second, even though  $R'/X$  may be small, it may not be small enough to assure convergence of

the perturbation expansion. Bisognano<sup>14</sup> has made *a priori* estimates of the radius of convergence which indicate validity of the expansion for

$$v_B R' L / g_0 < d_n / 2c(n), \quad (102)$$

where  $d_n$  is the spacing between the  $n$ th eigenvalue and its nearest neighbor and  $c(n)$  is a model-dependent number. If this limit is exceeded the perturbation expansion may continue to converge, but will give a deceiving description of the longitudinal motion.

Our final assumption was that the zero-resistance modes are nondegenerate. That mode degeneracy can be quite important can be seen with the standard impedance model for an unbunched beam. Our perturbation theory applied to either the mode  $\cos(kx)$  or  $\sin(kx)$  yields a slight damping rate. However, because  $\cos(kx)$  and  $\sin(kx)$  are degenerate, linear combinations are allowed, i.e.,  $\exp(\pm ikx)$ ; when our perturbation theory is applied to these modes we obtain the correct (large) growth rate.

It is unlikely that exact degeneracy will occur in any model of a finite bunch; thus, we must investigate the consequences of approximate degeneracy. Very long beams where modes become very close together must display some kind of "instability" and there are two ways that this can happen. The first possibility is that instability (nonzero imaginary part of  $\omega$ ) will set in at some small, but nonzero mode spacing (e.g., long but not infinite bunch). This could be seen in an exact (nonperturbative in  $R'/X$ ) dispersion analysis of our fourth-order model. Examination of a model problem,<sup>4</sup> however, seems to indicate that this does not happen. Thus, it is unlikely that a dispersion analysis of Eq. (90) will yield the behavior that must go over into the unbunched beam instability.

Another kind of "instability" can result if we consider an initial perturbation which is a superposition of nearly degenerate normal modes with large amplitudes which, at  $t = 0$ , nearly cancel to give the small initial perturbation. At a later time the different frequencies will give a different phase relationship between the modes, which then appears as growth. Of course, in a "beat period," i.e., approximately  $2\pi/\Delta\omega$  where  $\Delta\omega$  is the mode spacing, the phases will return to their initial values yielding a subsequent "decay." As the mode spacing  $\Delta\omega$  goes to zero (the limit of a uniform bunch) the subsequent decay never occurs, only growth is seen. What we have been describing crudely can be formulated mathematically by the statement that the uniform beam instability results from a bifurcation from the continuum.

The effects of such an apparent instability can be expected to be as destructive as those of a true instability. Thus it is important to try to determine an effective growth rate in order to compute the amplification in a "beat period"  $2\pi/\Delta\omega$ . To do this we consider a model with just two nearly degenerate modes.

If we assume that two modes are nearly degenerate, then we can compute an effective growth rate using Eq. (76). We now assume that  $I_n^{(0)}$  is a linear combination of two almost degenerate modes of different parity; we then find an effective growth rate

$$\gamma \sim \frac{1}{2|\omega_n|} \frac{\int I_n^{(0)} [v_p^2(x) v_B R' / g_0] (\partial I_n^{(0)} / \partial x)}{\int I_n^{(0)2}(x) dx} \quad (103)$$

Note that the numerator of this equation no longer vanishes because of the indefinite parity of  $I_n^{(0)}$ . Of course, there is a second "mode" that damps with the same rate. For a specific set of modes it is easy to evaluate Eq. (103), but a useful estimate is

$$\gamma \sim \alpha \pi v_p^2(0) v_B R' / \lambda g_0 \omega_n, \quad (104)$$

where  $\lambda$  is an effective "wavelength" of  $I_n$  and  $\alpha$  is a numerical factor that measures the mixing and overlap of the two modes; probably  $\lambda \sim 1/2$ . A crude estimate of the total amplification to be expected in this case is

$$n_{\max} / n_{\text{initial}} \sim e^{2\pi\gamma / \Delta\omega}. \quad (105)$$

Note that Eq. (104) goes over into the correct unbunched beam growth rate Eq. (2), if we let  $2\pi/\lambda = k$  and take  $\alpha = 1$ .

### B. Stability criterion

An approximate evaluation of the effective growth rate, Eq. (104), and the total amplification, Eq. (105), can be obtained using the fourth-order model. We take finite length effects into account by the prescription

$$k = 2\pi n / L, \quad (106)$$

where  $n$  is mode number and  $L$  is bunch length; the mode spacing given by Eq. (4.40) is, for small  $k$ , the constant

$$\Delta\omega = 2\pi v_p / L, \quad (107)$$

and for very large  $k$  is the constant

$$\Delta\omega = (6\pi/L)(T/M)^{1/2}. \quad (108)$$

For intermediate but large values of  $k$  we have

$$\Delta\omega_n \simeq [6n(2\pi)^2 / 2\omega_n L^2] (T/m). \quad (109)$$

We may estimate  $\gamma$  by Eq. (104). The condition  $2\pi\gamma / \Delta\omega < 1$ , for small amplification, becomes

$$T/m > L^2 v_p^2 v_B R' \alpha / 6ng_0 \lambda. \quad (110)$$

Note that this criterion involves  $R'(n)/n$  and thus is very frequency-dependent. For high frequencies, since  $R'(n)$  falls off quickly at high frequencies, the criterion is certainly satisfied. This is an important result, for it means that one will *not* have an effective instability of short-wavelength modes. If the criterion is not satisfied for a few low modes it is, of course, possible to handle them by feedback stabilization.

There are a number of improvements of these calculations which could be made. First one could, by solving the fourth-order model either exactly or approximately, refine the estimates for  $\Delta\omega$ . One could also refine the estimate for the effective growth rate. We suspect that the answer lies near the result, Eq. (103), since this goes over into the correct unbunched beam growth rate. These two improvements would result in a design criterion which would be reasonably accurate. Finally, to test all of these results one could numerically solve, as an initial value problem, the partial differential equation associated with the fourth-order model.

Bisognano, Haber, and Smith have numerically found threshold behavior for resistive instability.<sup>15</sup> They use the

modified impedance model, Eq. (46), and find a criterion of the form Eq. (102).

## VI. SUMMARY

In this paper we have derived, by two different methods, the integral equation which describes density oscillations in a warm particle bunch. We then analyzed this equation in perturbation theory—for small resistivity—and showed that in first order the motion is stable. This behavior is to be contrasted with that of a uniform beam where the motion is, in first order, unstable.

After performing a second-order calculation we devoted attention to the question of the validity of the perturbation theory approach. A high-order model, having a cut-off impedance and temperature effects included, is shown to be necessary in order to carefully study the density variation in a bunch. Using very crude estimates of the spectrum of modes in the absence of resistivity, we obtained a criterion for bunch stability. This criterion—which contains the impedance (as a function of frequency)—is presented as a design criterion for heavy-ion-fusion induction accelerators.

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