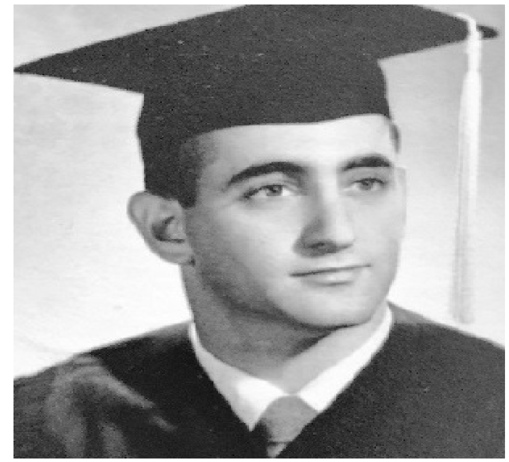
Martin Halpern Memorial Conference, March 29-30, 2019

"C-functions and C-theorems: From the Virasoro Master Equation to entanglement entropy.



Elias Kiritsis

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Prehistory

• In spring 1988 I was at the last year of my PhD.

• I got a call from Stanley Mandelstam who offered me a postdoc position at Berkeley. I was very excited and happy to accept it.

• Before coming to Berkeley I was working in CFT_2.

A BOSONIC REPRESENTATION OF THE ISING MODEL

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Received 18 June 1987

A bosonic construction of the critical Ising model is given. The stress–energy tensor has a different form from that of a free scalar field. Equivalence is established on an arbitrary compact Riemann surface. It is argued that this description is realized in a sector of the critical Askin–Teller model.

• I started generalizing this, and I obtained similar stress tensors with scalar exponentials for all G/H (coset) CFTs known at the time.

NON-STANDARD BOSONIZATION TECHNIQUES IN CONFORMAL FIELD THEORY*

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Received 22 November 1988

It is shown that G/H models can be constructed in terms of a number of free bosons with a stress energy tensor that contains vertex operators. Generalizations of this technique are also discussed.

• This was also worked out at the same time by Dunne+Haliday+Suranyi.

• I also realized that this could be done for any (2,0) operators available in a CFT, and derived a system of quadratic equations.

$$T_a(z) = \sum_{i=1}^{N} a_i \, \Phi_i(z) \quad , \quad \langle \Phi_i \Phi_j \rangle = G_{ij} \quad , \quad \langle \Phi_i \Phi_j \Phi_k \rangle = C_{ijk}$$

$$a_k = \sum_{i,j} a_i a_j \, C_{ijk} \quad \forall k \quad , \quad c = \sum_{i,j} G_{ij} a_i a_j$$

• Any solution T_a had a K-conjugate one $T - T_a$, and corresponded to a factorization of the CFT to two (commuting) parts $(T_a, T - T_a)$.

• Jeff Harvey and Lance Dixon were doing something similar (although they did not published it in the end)

• After discussing with Marty, he told me that he was thinking about something similar, and suggested to do something concrete: use the current bilinears present in any WZW CFT:

$$\Phi^{ab}(z) =: J^a(z) J^b(z) :$$

• Already the coset construction was well-known, and was a simple case of this more general setup. Moreover, it had given the biggest list of (solvable) CFTs known at the time (and even today).

Bardakci+Halpern, Goddard+Kent+Olive

HYSICAL REVIEW D VOLUME 3, NUMBER 10 15 MAY 1971

New Dual Quark Models*

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(Received 16 November 1970)

On the basis of new representations of the projective group, we construct some new dual quark models whose spin and internal symmetry are not multiplicative. One model is a factorized theory of exotic states with broken exchange degeneracy, ninth mesons being optional. The exotic states are suppressed three units below the Pomeranchon. In another model, with spin-orbit coupling and curved trajectories, both spin ghosts and orbital ghosts are involved in the Ward identities.

HYSICAL REVIEW D VOLUME 12, NUMBER 6 15 SEPTEMBER 1975

Quantum "solitons" which are SU(N) fermions*

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(Received 30 April 1975)

In two dimensions, we find a construction for an SU(N) quark field in terms of N real Bose fields. Hence, equivalence is shown between certain massive SU(N) Thirring models and systems of quantum sine-Gordon-type equations. From the point of view of the bosons, the "soliton-quark" SU(N) is topological. To minimize guesswork in the development of such correspondences, we employ a systematic blend of Mandelstam's operator approach with the interaction picture.

The Virasoro Master Equation (VME)

GENERAL VIRASORO CONSTRUCTION ON AFFINE g*

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Received 1 April 1989

We study the general Virasoro construction $L = L_{ab} {:}J^a J^b{:}$ on the currents of affine **g**, obtaining the master equation for the inverse inertia tensor L_{ab}. Sugawara and coset constructions are only the simplest solutions of this system, as illustrated here by a class of generalized spin-orbit constructions with generically irrational central charge.

- We had derived the equations that defined a Visaroso operator in the Hilbert space of G-WZW model.

- It was a rather pompous name for a set of quadratic equations.

The VME (II)

- The algorithm was simple: pick a group G and a level k (ie. a current algebra) characterized by, G^{ab} and $f^{ab}{}_{c}$.

- Consider the (2,0) operator

$$T_L = \sum_{a,b} L_{ab} \; : J^a J^b :$$

- If L_{ab} satisfies

$$L_{ab} = \sum_{cd,ef} C_{ab}{}^{cd;ef} L_{cd} L_{ef} \quad , \quad c = \sum_{a,b,c,d} G^{ab,cd} L_{ab} L_{cd}$$

Then T_L is a Virasoro operator with central charge c (generically irrational). The original theory can be factorized in a product: $(T_L, T_G - T_L)$

- The G/H coset models were a special case (solved by Mike Douglas in 1988).

- They could be written as an H-gauge theory of the G-WZW theory. But here you would need to do 2-d gravity.

Solutions and properties

• Although the system looks elementary, the generic number of solutions is enormous: In general one expects $2^{dimG(dimG-1)/2}$ solutions.

• A tiny number of them are symmetry related and a tiny number are rational.

• Already, on SU(3) we expect $\frac{1}{4} \times 10^9$ solutions.

• Some were found using special ansatze and then solving the reduced algebraic equations

• Infinite classes were found perturbatively in the level k by Marty and Niels, by mapping the problem to graph theories.

• The lowest irrational central charge found was on SU(3)

$$c_{\text{minimal}} = 2\left(1 - \frac{1}{\sqrt{61}}\right) \simeq 1.7439$$

• Exceptionally, there are continuous manifolds of solutions corresponding to marginal deformations of CFT's.

• There is a special subclass of solutions (self K-conjugate solutions) where $c = \frac{c_G}{2}$. Such theories have rational c but irrational conformal weights.

• It turns out that they determine the Witten Index in Morse theory (later).

• Conformal weights can be found by brute force diagonalization of L_0 on affine representations. Technically difficult to do except a few subcases.

• No technique is known to obtain correlators in the irrational cases.

• This is by far the largest class of CFTs (that are known to exist) in two dimensions.

Zamolodchikov's C-theorem

• A couple of year earlier Zamolodchikov made a remarkable discovery in two dimensions which took some time to reach the West:

"Irreversibility" of the flux of the renormalization group in a 2D field theory

A. B. Zomolodchikov
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(Submitted 20 May 1986)
Pis'ma Zh. Eksp. Teor. Fiz. **43**, No. 12, 565–567 (25 June 1986)

There exists a function $c(g)$ of the coupling constant g in a 2D renormalizable field theory which decreases monotonically under the influence of a renormalization-group transformation. This function has constant values only at fixed points, where c is the same as the central charge of a Virasoro algebra of the corresponding conformal field theory.

• He showed that in 2d QFTs, there is a function $C(g_i)$ that :

(a) Is decreasing along an RG trajectory (non-perturbative)

(b) Is extremal at fixed points (perturbation theory).

(c) Is equal to the Virasoro central charge at fixed points (non-perturbative).

(d) In perturbation theory the β-functions are gradients of the C-function:

$$\beta^i(g) = -\frac{1}{12} G^{ij} \frac{dC}{dg^j}$$

The C-theorem follows from

$$\frac{dC}{dt} = \frac{dC}{dg^i}\ \frac{dg^i}{dt} = -12\beta^j\ G_{ij}\ \beta_i < 0$$

Some definitions for further use:

• "weak" C-theorem: $c_{UV} \geq c_{IR}$.

• Existence of a C-function: A continuous function that interpolates between CFT_{UV} and CFT_{IR} and is decreasing along the flow.

• Strong C-Theorem: $\beta^i \sim G^{ij} \frac{\partial C}{\partial g^j}$

♠ In 2d, we have the first two, non-perturbatively and the third perturbatively.

♠ It is a folk theorem that the strong form excludes limit cycles in the RG. Curtright and Zachos pointed however a loop hole if the β-functions have branch cuts.

♠ In 2d limit cycles cannot happen (in unitary QFTs) because scale invariance and locality implies conformal invariance.

Todorov, Polchinski

♠ Despite many recent efforts a similar question is open in 4d.

Generalizations

• In 4 dimensions, it was conjectured and proven that the correct C-quantity is the a-anomaly

Cardy, Komargodski+Schwimmer

• This is the weak form of the C-theorem: $a_{UV} \geq a_{IR}$

• The strong form is known to be valid in perturbation theory.

Osborn

• There are two distinct types of β-functions

$$T_\mu{}^\mu = \sum_i \beta_i \; O_i \quad , \quad \frac{dg_i}{dt} = \tilde{\beta}^i \quad , \quad \delta S \equiv \sum_i g_i \int O_i$$

• They are related in perturbation theory. The strong version of the a-theorem can be proven only in perturbation theory

Jack+Osborn

• It is not yet known if scale invariance implies always conformal invariance.

• It has been found that in holographic RG-flows, the holographic β-functions can have branch points in which the second derivative diverges.

Kiritsis+Nitti+Silva-Pimenta

• It is this behavior that Curtright and Zachos suggested can lead to limit cycles although the strong-version of the C-theorem is valid.

• It could be proven however, that no limit cycles were allowed in holographic theories.

Kiritsis+Nitti+Silva-Pimenta

Back to two dimensions

• With Marty, Niels and Amit in 1989 we made a simple observation: by postulating a flow on "Affine Virasoro space" we could obtain all the ingredients of the C-theorem.

EXACT C-FUNCTION AND C-THEOREM ON AFFINE-VIRASORO SPACE

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Received 28 December 1990

An exact C-function, which is an action for the Virasoro master equation, is obtained on affine-Virasoro space. The solutions of the master equation are fixed points of an associated flow, which obeys a C-theorem. The closed sub-flow $SO(n)_{diag}$ is a flow on the space of graphs, and the associated Morse polynomials are known generating functions in graph theory. The general flow also implies a method for prediction of sporadic conformal deformations. We note the resemblance of this system to the expected form of an exact renormalization group equation.

• We could find an explicit C-function

$$C(L) = 6\ L_{ab}\ G^{ab;cd}\ L_{cd} - 2L^{ab}L^{cd}L^{ef}C_{ab;cd;ef}$$

so that:

$$\frac{\partial C}{\partial L_{ab}} = -12G^{ab;cd}\ \beta_{cd}(L) \quad , \quad \beta_{ab} = L_{ab} - C_{ab}{}^{cd,ef}L_{cd}L_{ef}$$

• $\beta_{ab}(L^*) = 0$ is the condition of of existence of the CFT(L^*) ($T_L \equiv L^*_{ab} : J^a J^b$: satisfies the Virasoro algebra).

$$C(L^*) = c = 2L^*_{ab}G^{ab;cd}L^*_{cd}$$

• If we now postulate a flow:

$$\frac{dL_{ab}}{dt} = \beta_{ab}(L)$$

then along this flow, a C-theorem follows:

$$\frac{dC}{dt} = \frac{\partial C}{\partial L_{ab}}\ \frac{dL_{ab}}{dt} = -12\beta_{ab}\ G^{ab,cd}\ \beta_{cd} \leq 0$$

• The quadratic part of C corresponds to the "classical" C-function contribution, while the cubic part is "one-loop". From this point of view, The C-function on Affine Virasoro space is "one-loop exact".

The scaling regime

• Near a solution L^* (=CFT) the flow is characterized by the eigenvalues of the stability matrix: $\frac{\partial^2 C}{\partial L^2}$.

• This is the analogue of the $d - \Delta_i$ of the standard RG.

• In this case we could characterize all the eigenvalues of the stability matrix of the theory with T_{L*} as follows:

1. There is a universal eigenvalue +1 with eigenvector L^* and -1 with eigenvector $L_G - L^*$

2. There are dim G-dim h zero eigenvalues that are related to the group symmetry.

3. The rest are given by $1 - \Delta_{ab}$ where Δ_{ab} are the chiral dimensions of the operators $: J^a J^b :$ in the theory L^*.

Applications

• Although the Affine Virasoro (AV) flows were ad hoc, they were very useful in organizing the solutions of the VME

• The solutions were points (and sometimes manifolds) in

$$\mathbb{R}^{DimG(DimG-1)/2}$$

• The AV flow put an order in this manifold of fixed points, by introducing both a "height function" and a flow.

• From the flow and perturbation theory we predicted the existence of marginal manifolds of solutions knowing only the perturbative data.

• We can apply Morse theory to C by constructing the Morse polynomial

$$M(t,G_k) \equiv \sum_{\text{fixed points}} t^{N_r}$$

with N_r the number of relevant directions of each fixed point.

• In a subset of cases this polynomial can be computed.

Odd dimensions

• In odd dimensions there is no conformal anomaly.

• It was proposed in $d = 3$ that the analogue of the central charge is played by the renormalized partition function on the sphere for CFTs.

Jafferis, Jafferis+Klebanov+Pufu+Safdi

• But the associated partition function fails to be a monotonic F-function along the the flow.

Klebanov+Pufu+Safdi, Taylor+Woodhead

• An alternative "F-function" was proposed: the appropriately renormalized entanglement entropy associated to an S^2 in $\mathbb{R}^3$.

Myers+Sinha, Liu+Mazzei

• There is a general proof that in 3d this is always monotonic.

Casini+Huerta+Myers, Casini+Huerta

• A general analysis of the RG flows of 3d QFTs on S^3 has shown that:

Ghosh+Kiritsis+Nitti+Witkowski

a. There are several other F-functions that can be defined from the partition function that are monotonic along RG Flows in 3d.

b. Some of them, although defined from the partition function, coincide with the entanglement entropy on dS_3.

c. This bridges, the gap between the partition function and entanglement entropy.

• This is extendable to 4 and more dimensions, and is currently under investigation.

Outlook

• The space of CFTs which are solutions of the VME, are the largest set of CFTs in any dimension we know exist.

• This is a problem that Marty worked on for at least 20 years and is part of his legacy.

• The problem however is not fully solved.

• This space of CFTs is a good arena to test our tools on QFTs beyond perturbation theory.

• Several questions seem still interesting:

♠ Develop a 2d gravity description of the generic AF CFT.

♠ Find efficient ways to calculate their (low-lying) spectra and correlators.

♠ Derive physically the AF flow.

• Marty, will be remembered by all of us, as long as we live

• But his impact in Physics will outlive all of us.

Detailed plan of the presentation

- Title page 0 minutes
- Prehistory 5 minutes
- The Virasoro Master Equation 6 minutes
- VME (II) 8 minutes
- Solutions and properties 11 minutes
- The Zamolodchikov C-Theorem 15 minutes
- Generalizations 17 minutes
- Back to two dimensions 20 minutes
- The scaling Regime 22 minutes
- Applications 23 minutes
- Odd dimensions 25 minutes
- Outlook 26 minutes